Relational Refinement Types for Higher-Order Shape Transformers

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Joint work with Gowtham Kaki

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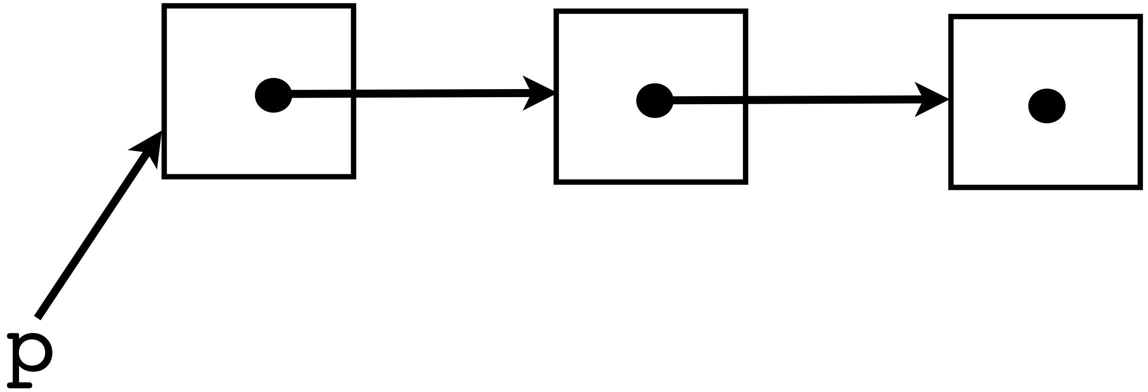
Shape Analysis

In imperative settings, shape analysis is concerned with discovering/verifying the shape of a pointer into memory

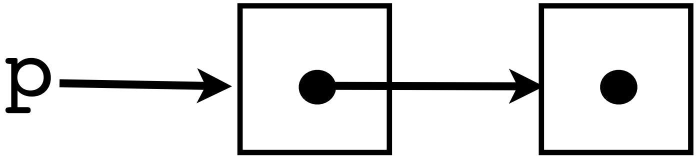


```
p = LinkedList
```

Shape Analysis for Functional (Typed) Programs

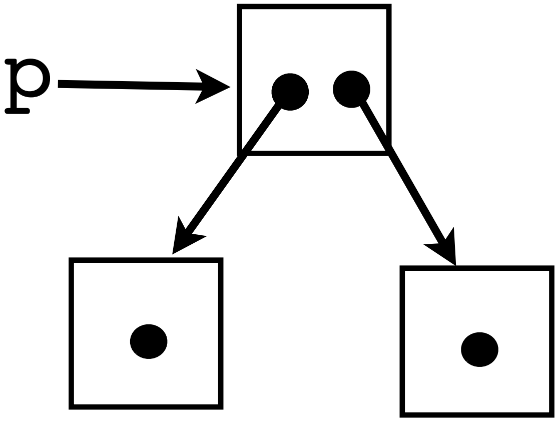
In functional languages, we have have types

```
p = Cons(.,Cons(., Nil))
```



p : α list

```
p = B(B(L,.,L),.,B(L,.,L)
```



p : α tree

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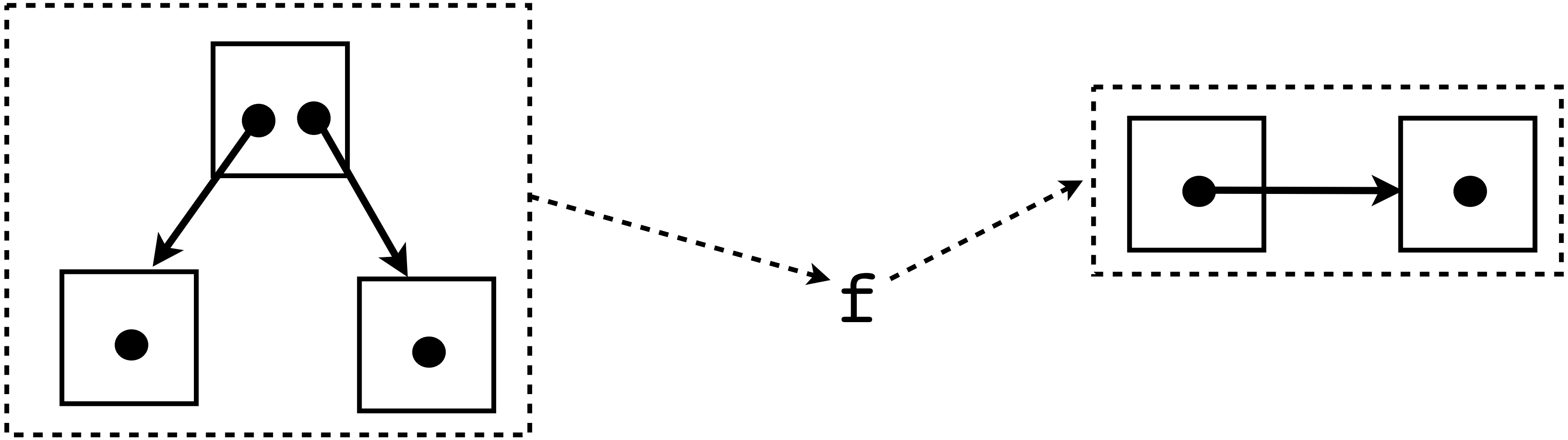

$$f : \alpha\ \texttt{tree} \longrightarrow \alpha\ \texttt{list}$$

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How can we use types to express precise shape information?

$$\texttt{f : \{t:}\alpha\texttt{ tree\}} \longrightarrow \texttt{\{l:}\alpha\texttt{ list}|\varphi\texttt{\}}$$

$$\underbrace{\varphi}_{\text{type refinement predicate}} \Leftrightarrow \texttt{SomeShape(l)}\equiv\texttt{SomeOtherShape(t)}$$

Reasoning about shapes

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    - ✦ E.g., length, sorted-ness, height, balance, membership, ordering, dominance, symmetry, etc.
  - ★ Polymorphism and higher-order functions

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  - ★ Polymorphism and higher-order functions
- Tension
  - ★ desire expressive specifications over the shape of data
  - ★ but want automated verification of their correctness

Example

```
rev : {l : ’a list} ⟶ {ν: ’a list | ν = rev’(l)}

fun rev [] = []
  | rev x::xs = concat (rev xs) [x]
```

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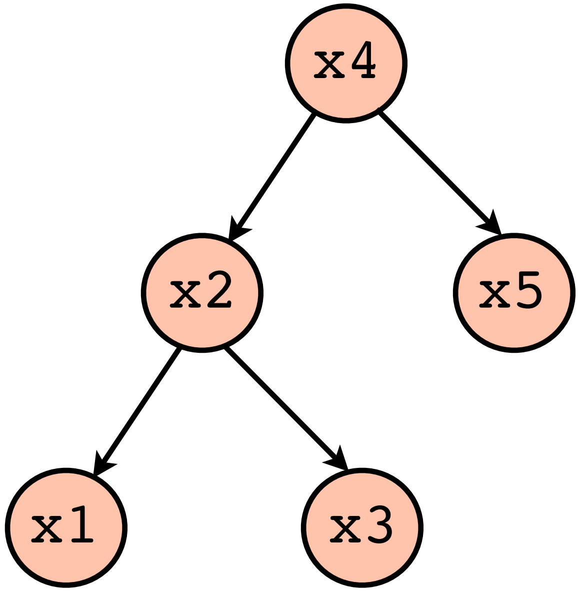
We want

- ★ To reason structurally about the order of elements in the list
- ★ Without appealing to an operational definition of how that ordering is realized

Example



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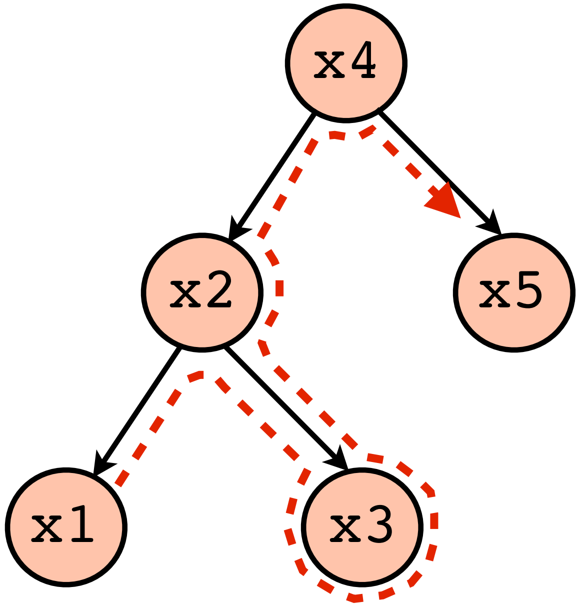
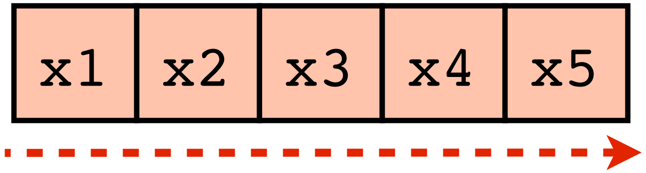
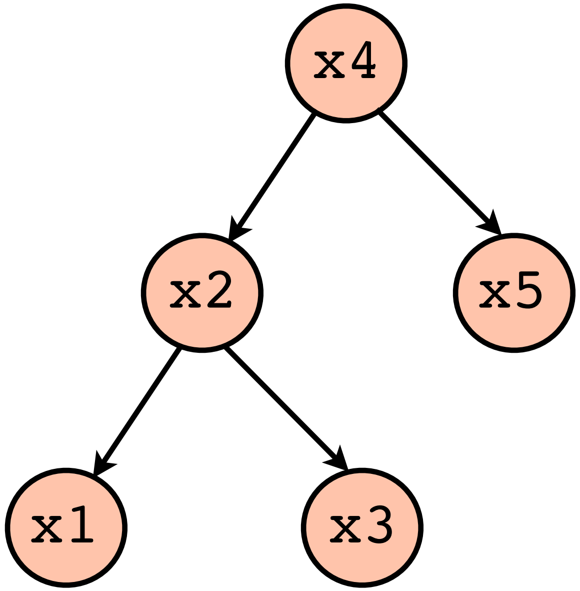




inOrder : {t:α tree} ⟶ {l:α list|φ}

φ ⇔ forward-order(l)=in-order(t)

Post-Order



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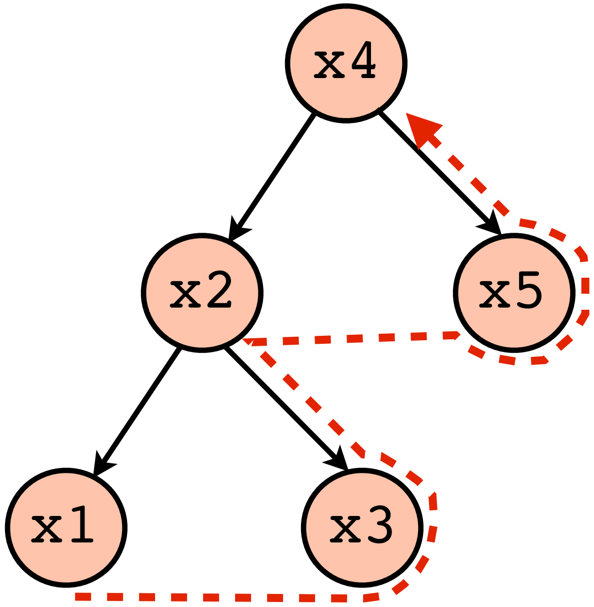

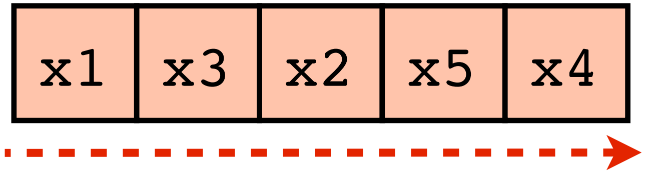

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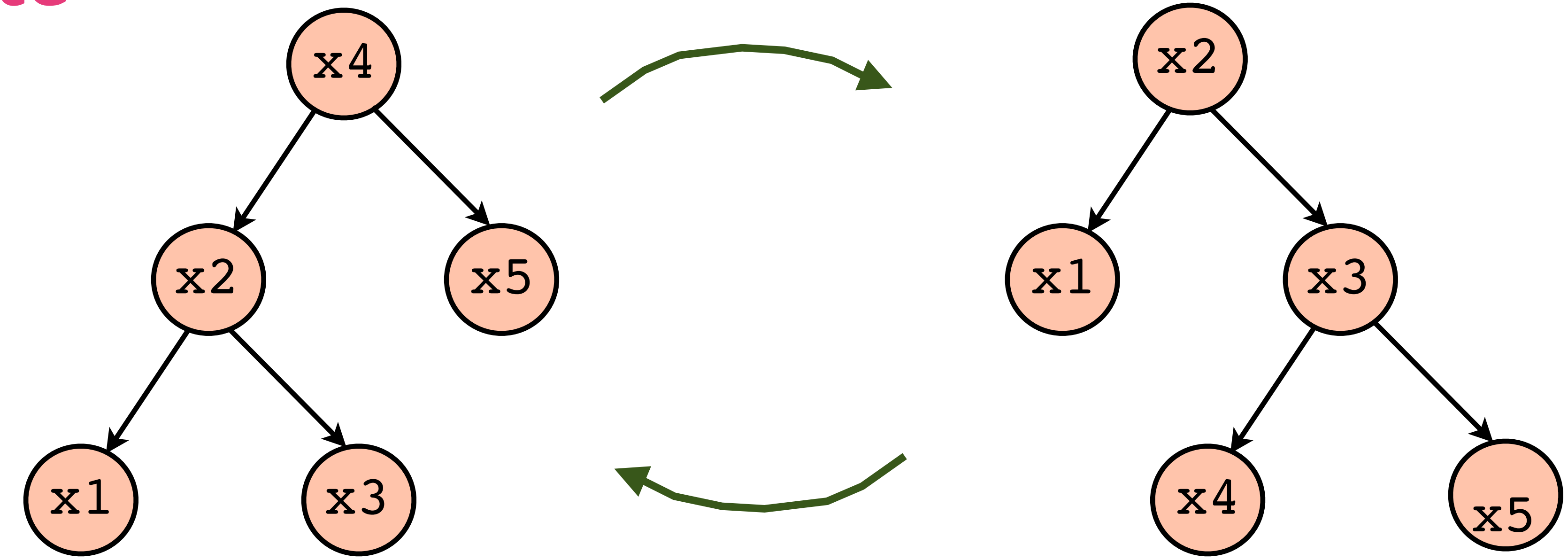
Rotate



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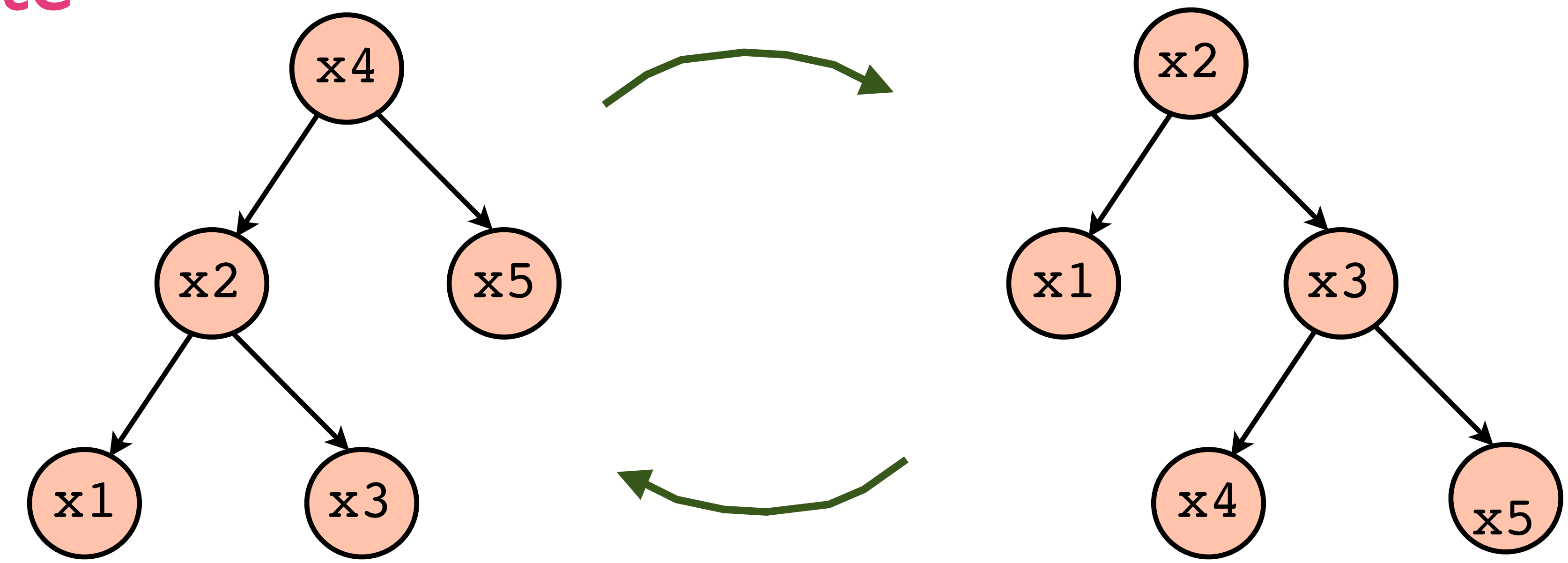


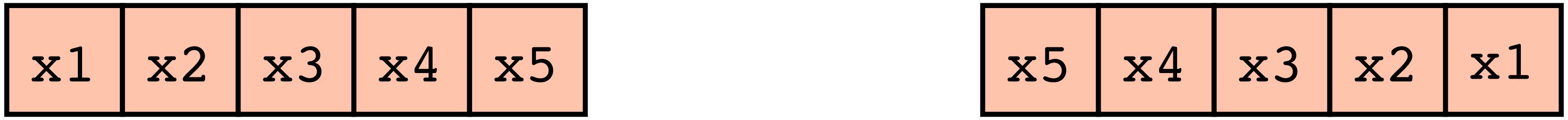
x1	x2	x3	x4	x5

x1	x4	x5	x3	x2

rotate : {t1:α tree} ⟶ {t2:α tree | φ}

φ ⇔ in-order(t1)= post-order(t2)

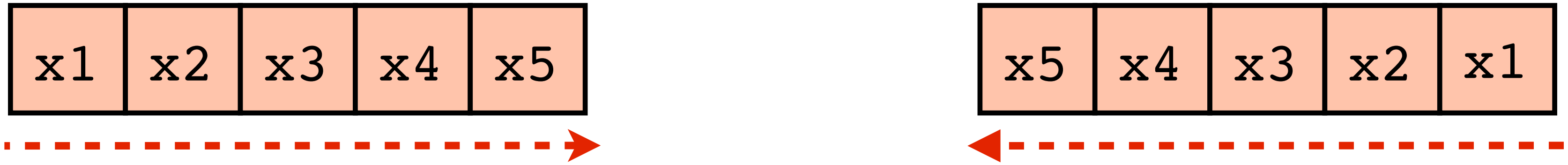
Reverse



x1	x2	x3	x4	x5

x5	x4	x3	x2	x1

Reverse



$$\mathtt{rev} : \{\mathtt{l1}{:}\alpha\ \mathtt{list}\} \longrightarrow \{\mathtt{l2}{:}\alpha\ \mathtt{list} \mid \varphi\}$$

$$\varphi \Leftrightarrow \mathtt{backward\text{-}order(l2)=forward\text{-}order(l1)}$$

We need ...

Type refinements (φ) to be predicates over an expressive language.

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Should serve as a common medium to express fine-grained shapes of data structures, such as `in-order`, `pre-order`, `post-order`, `forward-order`, and `backward-order`

Observe ...

What is common among `pre-order`, `post-order`, `forward-order`, and `backward-order`?

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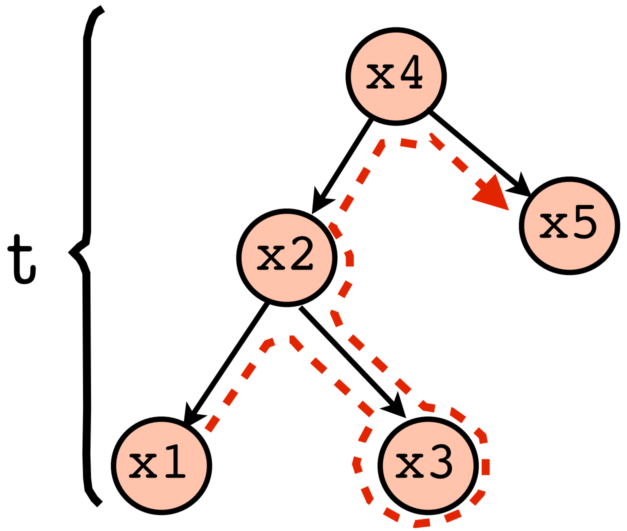
Expressible as binary relations

For Example ...



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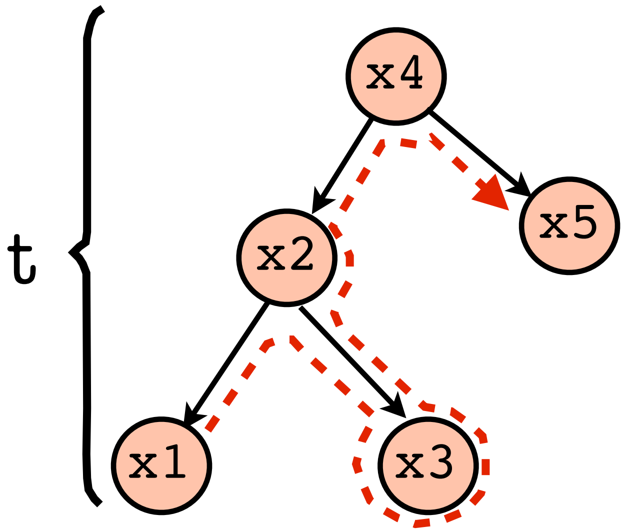

in-order of t is binary relation such that: in-order(x_i,x_j) ⇔ $i \leq j$

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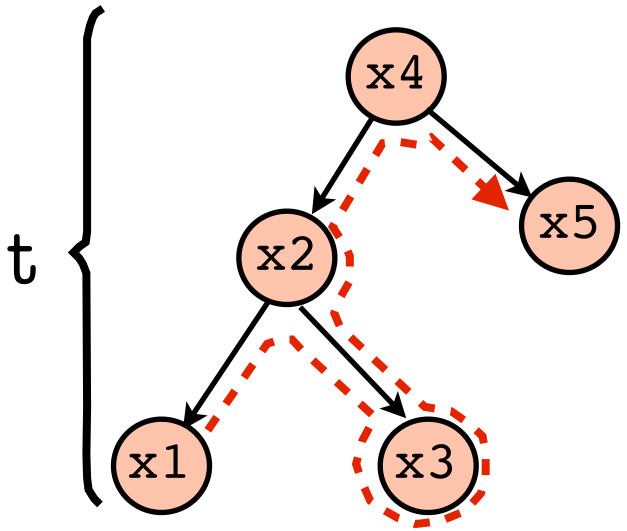

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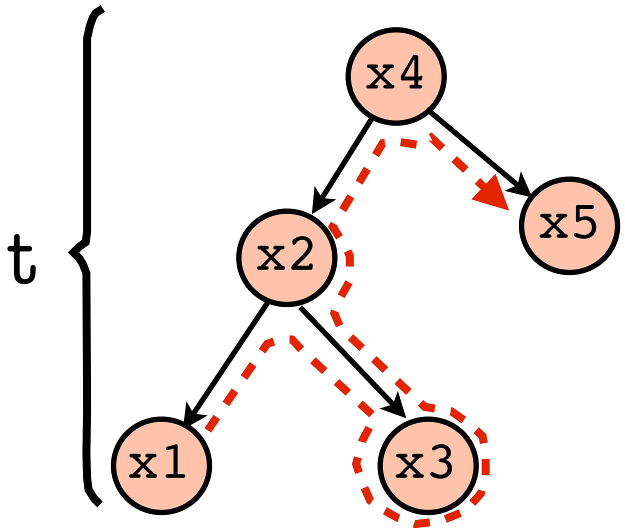

`in-order` of t is binary relation such that: `in-order`$(x_i,x_j) \Leftrightarrow i \leq j$

$$R_{io}(t) = \{(x_i,x_j) \mid i \leq j\}$$

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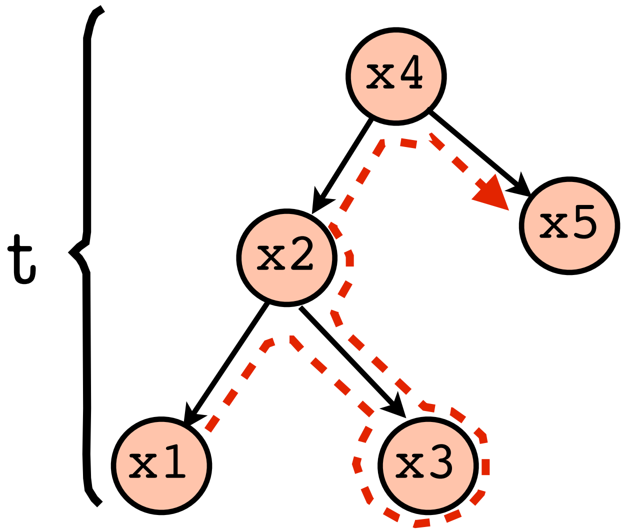

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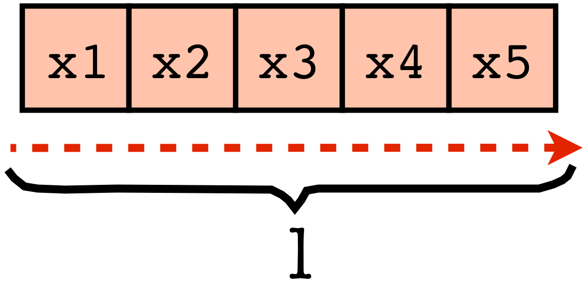

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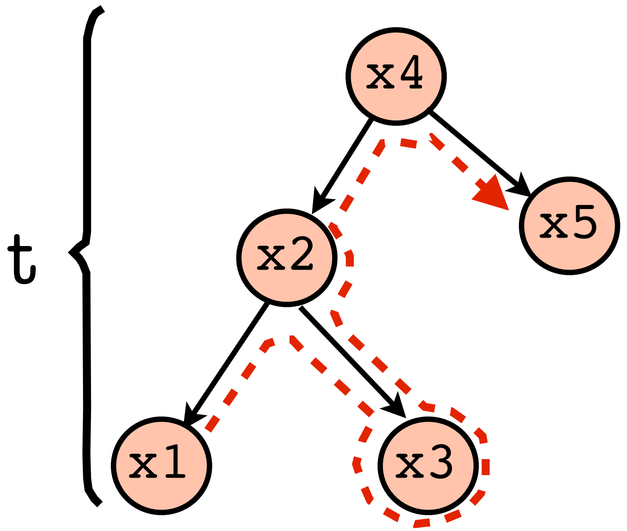

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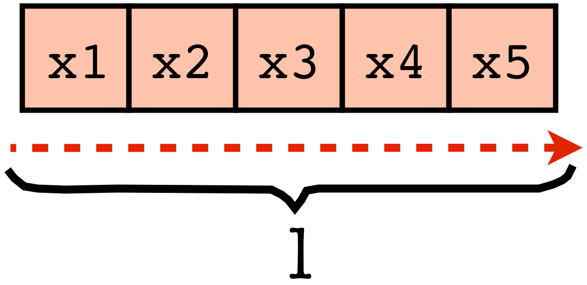

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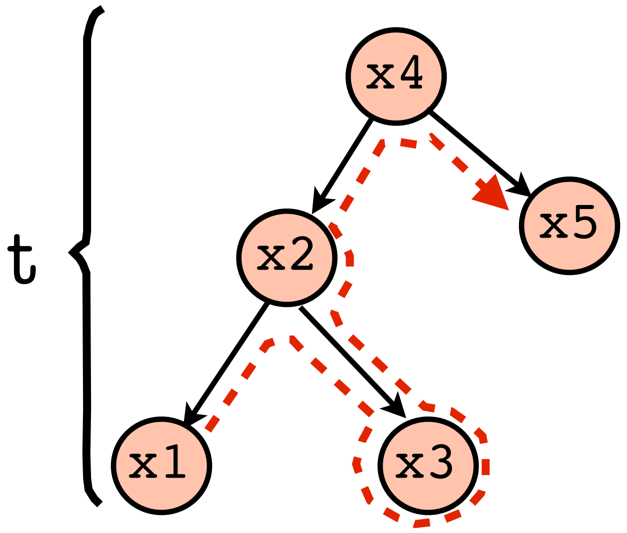

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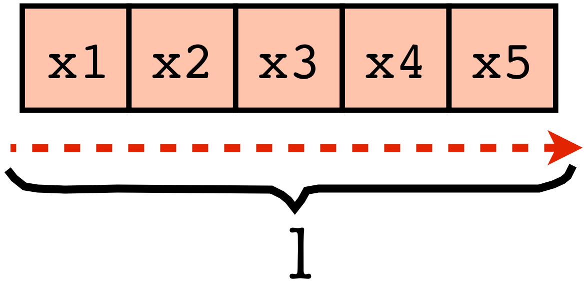

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$R_{fo}(l)$

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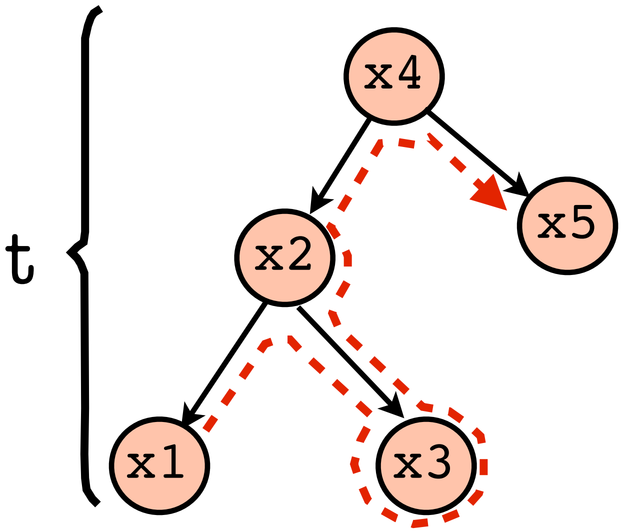

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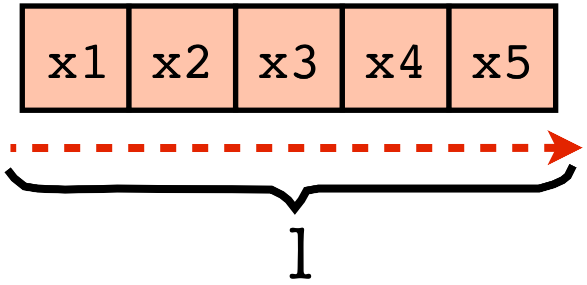

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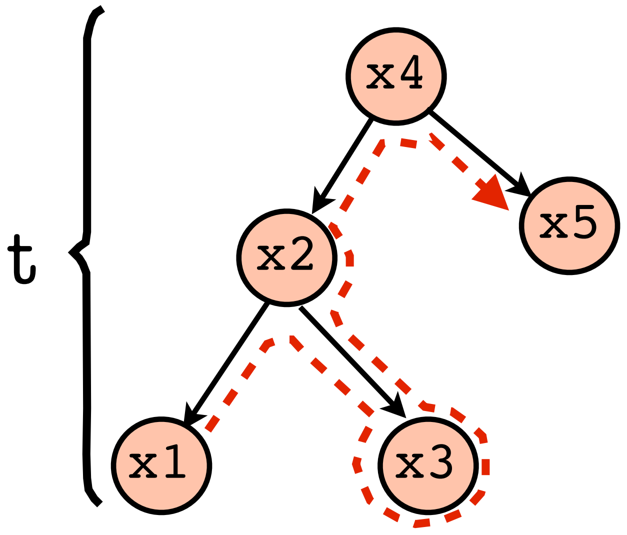

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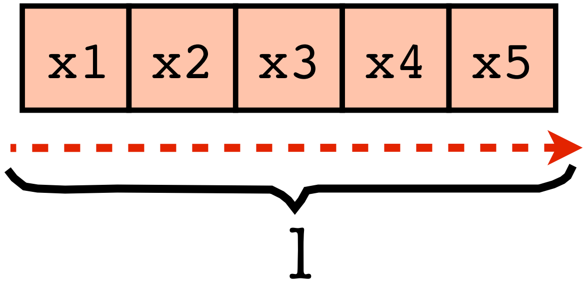


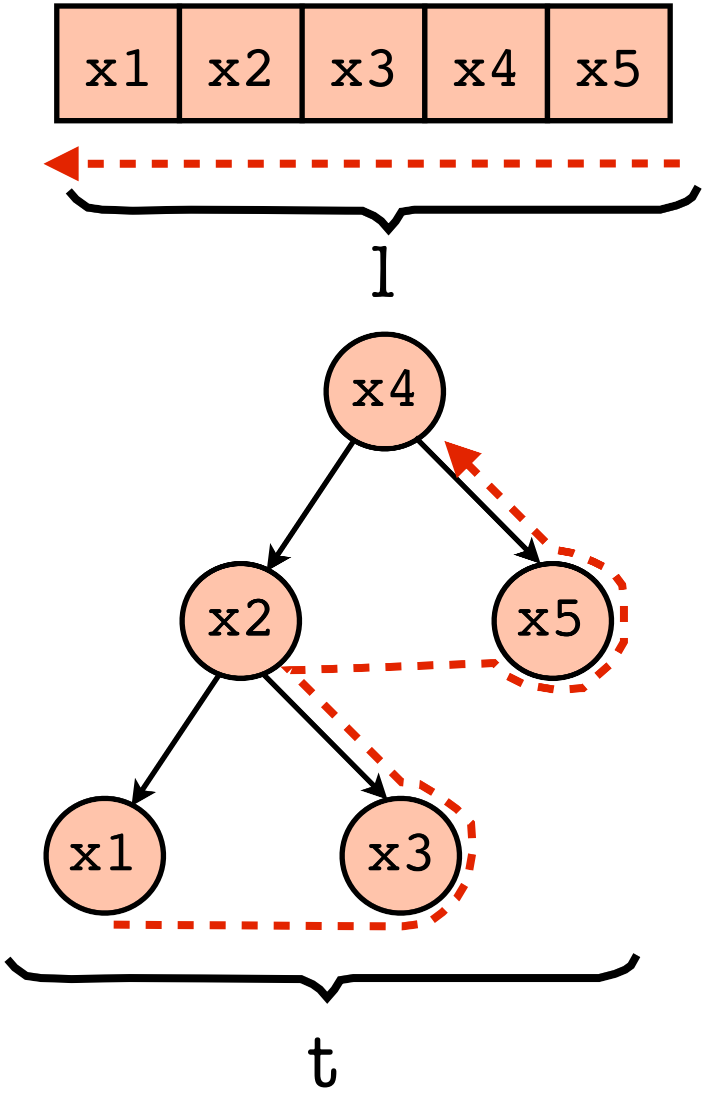
fwd-order of l is binary relation such that: $\mathtt{fwd\text{-}order}(x_i,x_j) \Leftrightarrow i \leq j$

$$R_{fo}(l) = \{(x_i,x_j) \mid i \leq j\}$$

⇒ If list l contains elements of tree t in pre-order, then

$$R_{fo}(l) = R_{io}(t)$$

More Relations



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Of supplementary value are unary membership relations:

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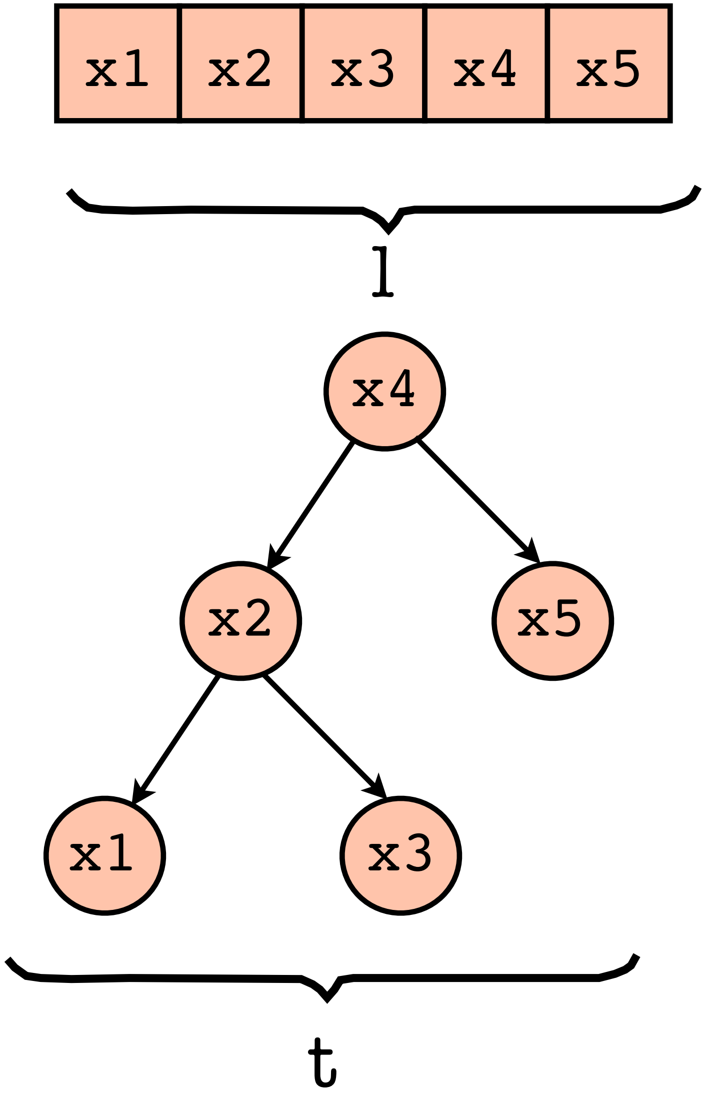
$R_{tm}(t) = R_{lm}(l) = \{x1, x2, x3, x4, x5\}$

`list-members`



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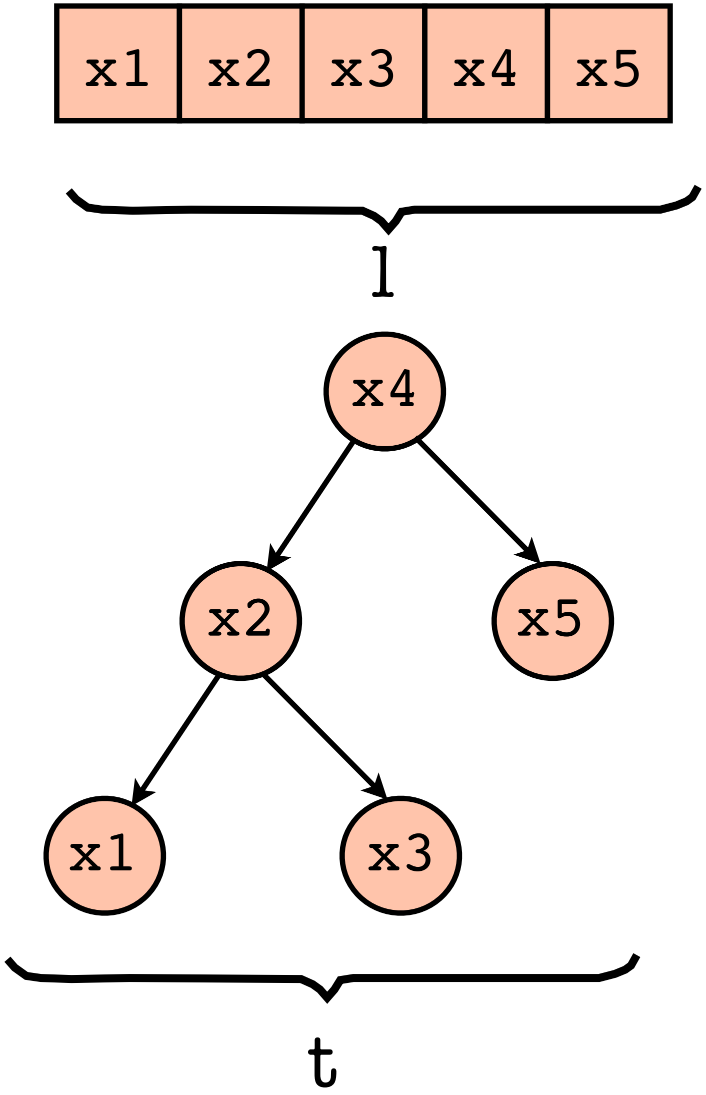

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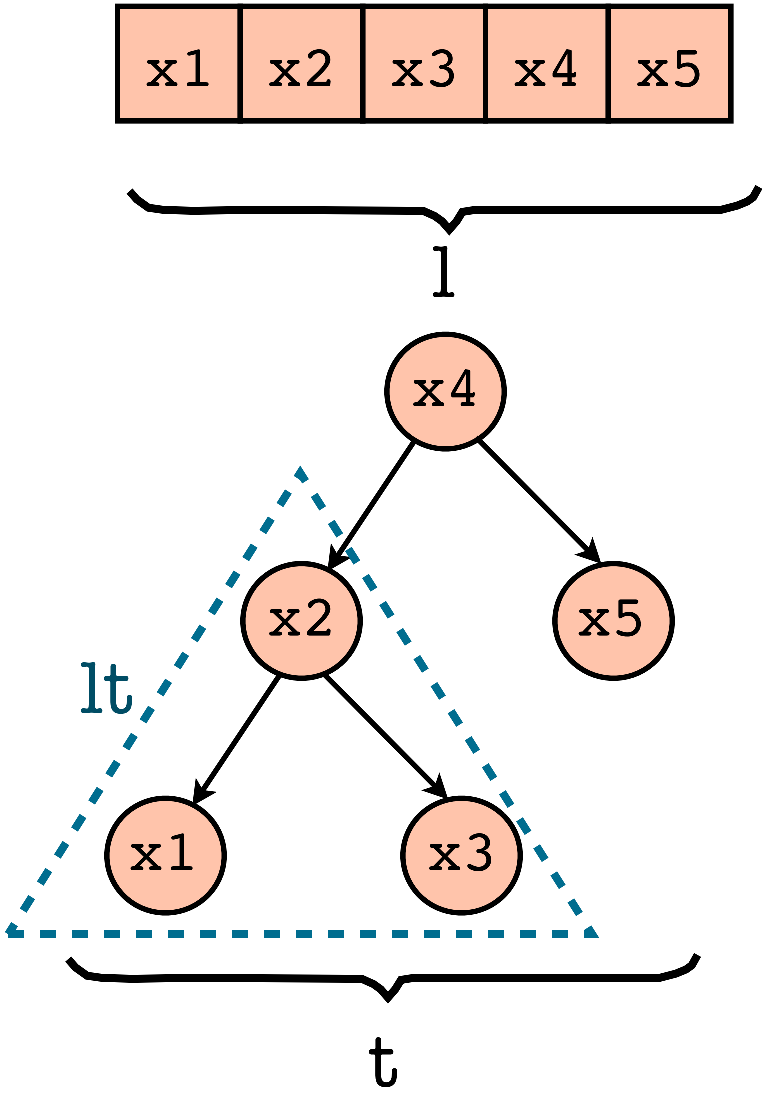

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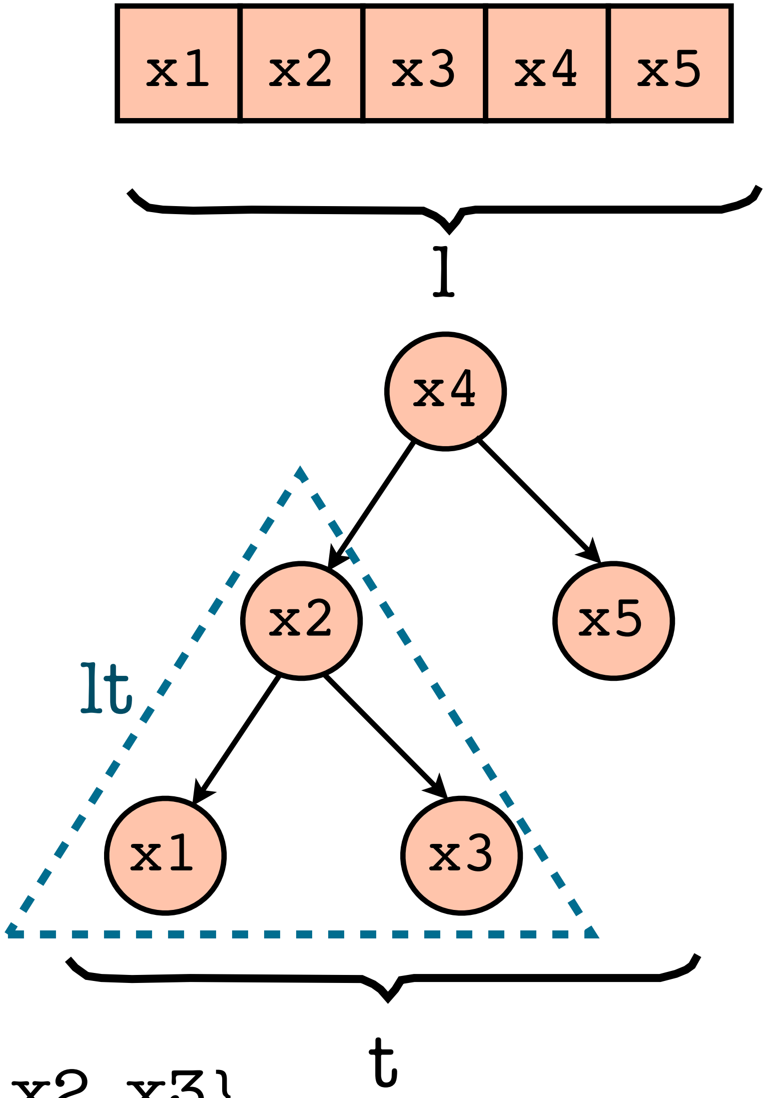

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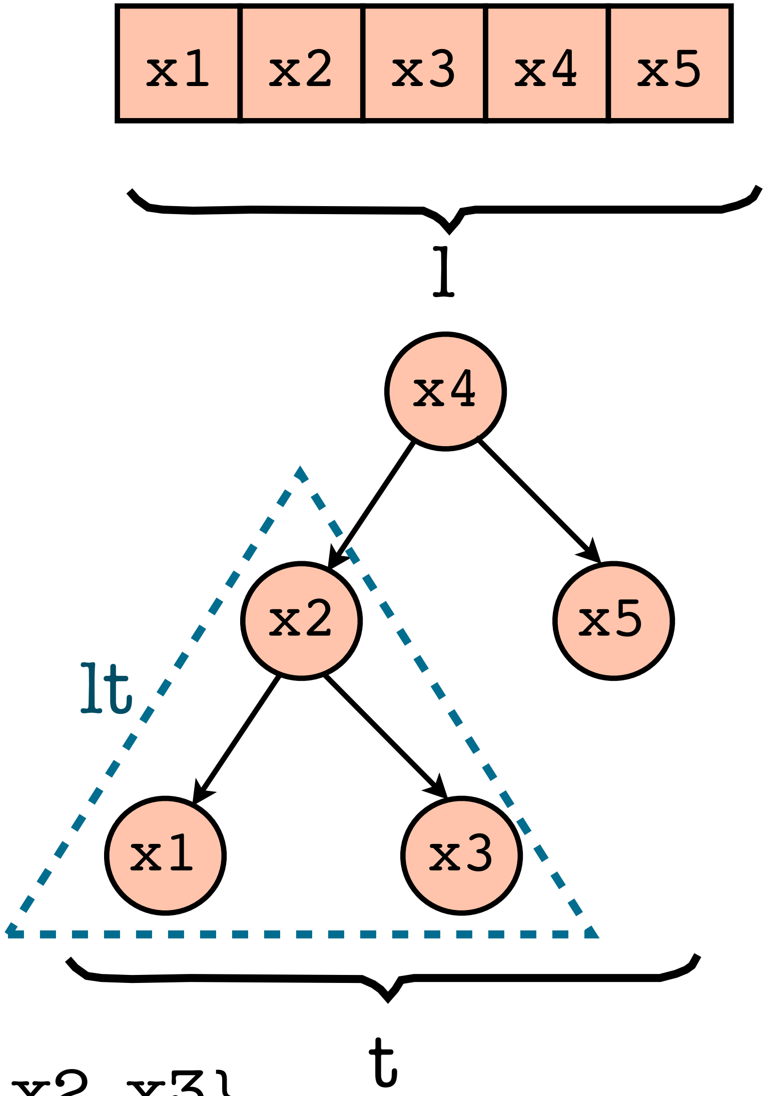

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They let us write assertions over binary relations like R_{po}

$$R_{tm}(lt) \times \{x4\} \subset R_{io}(t)$$

The Language of Relations ...

... with relational operators, such as union and cross-product, is capable of expressing fine-grained shapes.

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$$\texttt{relation } \texttt{R}_{\texttt{io}}\texttt{(Tree(L,n,R))} \ =$$

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$$\texttt{inOrder : \{t:}\alpha\texttt{ tree\}} \longrightarrow \texttt{\{l:}\alpha\texttt{ list} \mid R_{\mathrm{fo}}(\mathrm{l}) = R_{\mathrm{io}}(\mathrm{t})\}$$

$$\texttt{tail : \{l:}\alpha\texttt{ list\}} \longrightarrow \texttt{\{v:}\alpha\texttt{ list} \mid R_{\mathrm{fo}}(\mathrm{v}) \subset R_{\mathrm{fo}}(\mathrm{l})\}$$

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Relational types for polymorphic and higher-order functions must be general enough to relate different shapes at different call sites.

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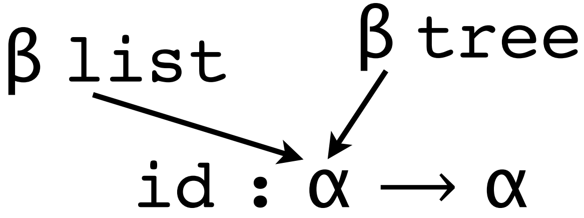
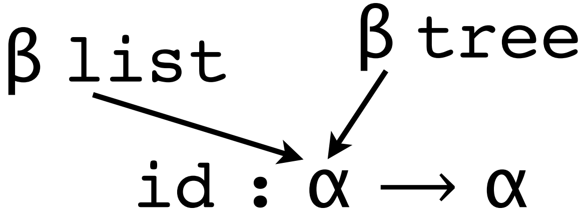


β `list`

β `tree`

`id : α ⟶ α`

`id` can take arguments of unknown shape



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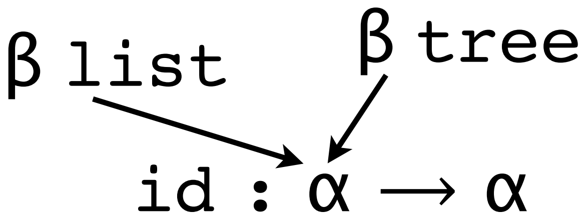
Shape of the argument is also the shape of its result

```
id : {x:α} ⟶ {y:α | Shape(y) = Shape(x)}
```

Relational Parameters



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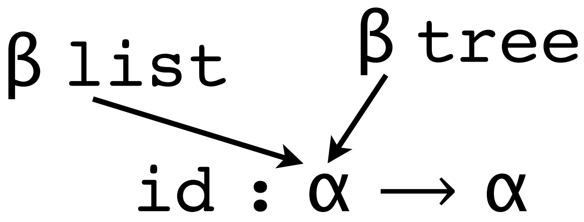
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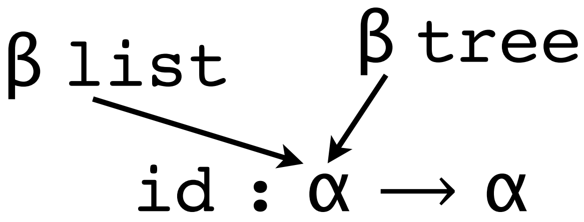
$$\texttt{id} : \{\texttt{x}:\alpha\} \longrightarrow \{\texttt{y}:\alpha \mid \underbrace{\texttt{Shape}}_{\rho}(\texttt{y}) = \texttt{Shape}(\texttt{x})\}$$

Relational Parameters



β list

β tree

id : α ⟶ α

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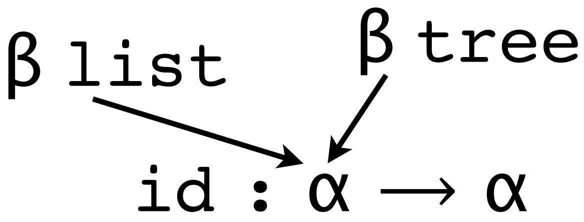

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$$\texttt{id : \{x:}\alpha\texttt{\}} \longrightarrow \texttt{\{y:}\alpha \mid \underbrace{\texttt{Shape}}_{\rho}\texttt{(y) = Shape(x)\}}$$

$$(\rho)\,\texttt{Id : \{x:}\alpha\texttt{\}} \longrightarrow \texttt{\{y:}\alpha \mid \rho\texttt{(y)} = \rho\texttt{(x)\}}$$

Relationally parametric type of id

A Parametric Type of pairMap ...

... by focusing on possible shape invariance between α and β

$$(\rho_\alpha,\rho_\beta)\ \texttt{pairMap} : \{x_1{:}\alpha\}*\{x_2{:}\alpha\}$$

$$\longrightarrow (\{x{:}\alpha\} \longrightarrow \{y{:}\beta \mid \rho_\beta(y) = \rho_\alpha(x)\})$$

$$\longrightarrow \{y_1{:}\beta \mid \rho_\beta(y_1) = \rho_\alpha(x_1)\}$$
$$* \{y_2{:}\beta \mid \rho_\beta(y_2) = \rho_\alpha(x_2)\}$$

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gets propagated to result type

A Parametric Type of pairMap ...

$$
\begin{aligned}
(\rho_\alpha,\rho_\beta)\ \texttt{pairMap} :\ & \{x_1{:}\alpha\} * \{x_2{:}\alpha\} \\
& \rightarrow (\{x{:}\alpha\} \rightarrow \{y{:}\beta \mid \rho_\beta(y) = \rho_\alpha(x)\}) \\
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& \quad * \{y_2{:}\beta \mid \rho_\beta(y_2) = \rho_\alpha(x_2)\}
\end{aligned}
$$

For eg:

$$\underbrace{(l_1, l_2)}_{\alpha\ \texttt{lists}} = \texttt{pairMap}\ (R_{io}, R_{fo})\ \underbrace{(t_1, t_2)}_{\alpha\ \texttt{trees}}\ \texttt{inOrder}$$

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For eg:

$$\underbrace{(\texttt{l}_1, \texttt{l}_2)}_{\alpha\ \texttt{lists}} = \texttt{pairMap}\ (\mathcal{R}_{io}, \mathcal{R}_{fo})\ \underbrace{(\texttt{t}_1, \texttt{t}_2)}_{\alpha\ \texttt{trees}}\ \texttt{inOrder}$$

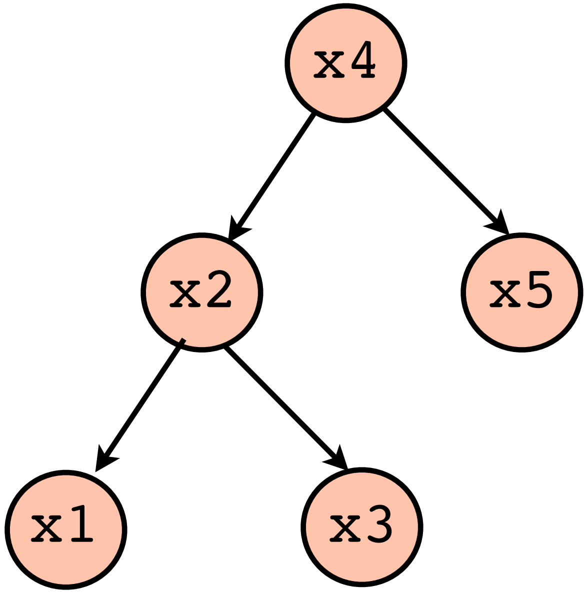
explicit instantiation
of
relational parameters

treefoldl

```
treefoldl f i (Node n) = f i n
        | f i (Tree left node right) =
              treefoldl f (f (treefoldl f i left) node) right
```

```
val inOrder = fn t => treefoldl t []
      (fn acc => fn x => acc ++ [x])
```

inOrder t =



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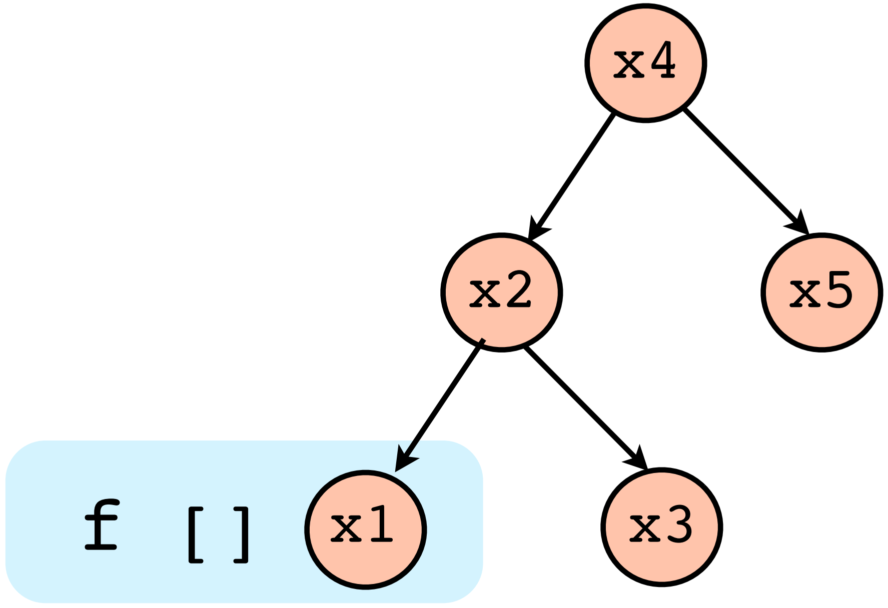

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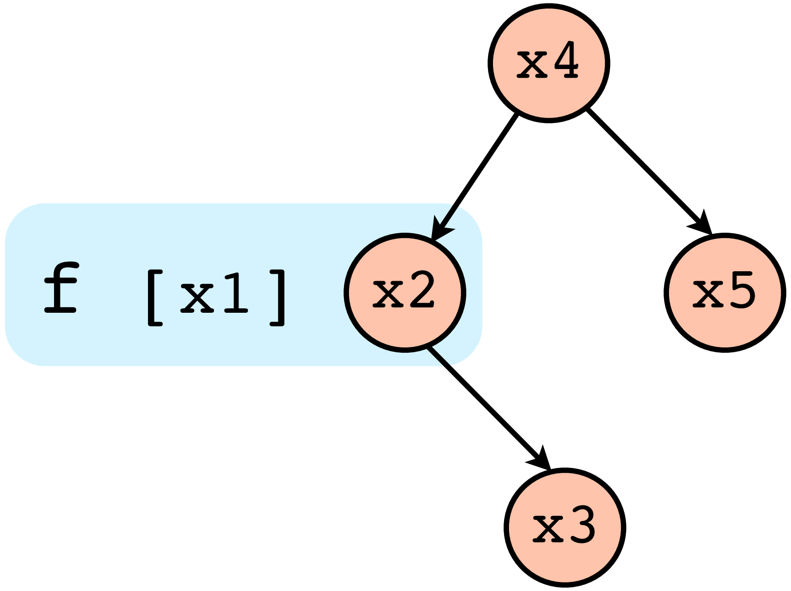

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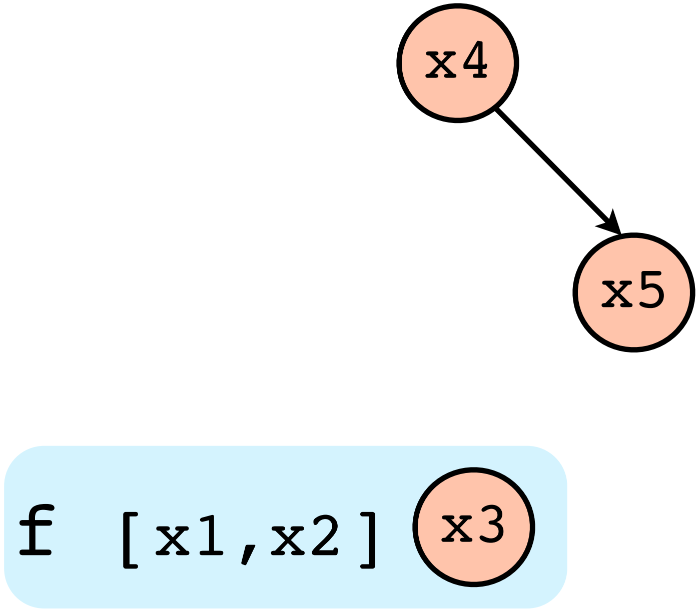

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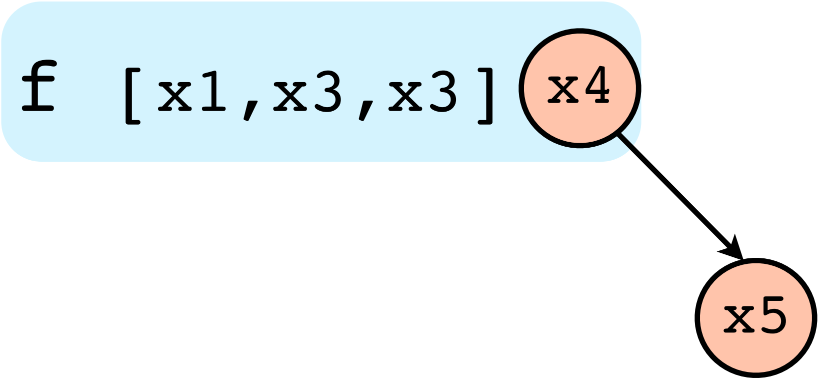

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val inOrder = fn t => treefoldl t []
      (fn acc => fn x => acc ++ [x])
```

```
inOrder t =        f [x1,x3,x3,x4] x5
```

treefoldl

```
treefoldl f i (Node n) = f i n
        | f i (Tree left node right) =
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val inOrder = fn t => treefoldl t []
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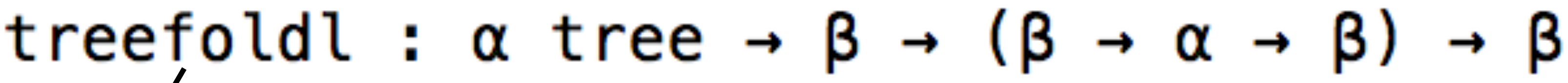
```

inOrder t = [x1,x3,x3,x4,x5]

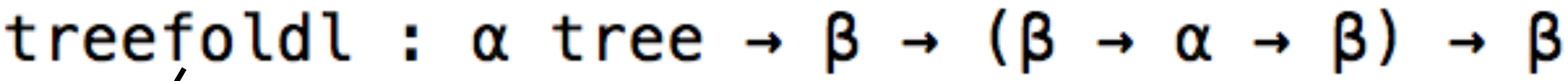
treefoldl



```
treefoldl : α tree → β → (β → α → β) → β
```

folds a tree from left to
right in in-order

treefoldl



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A parametric type can be constructed to relate `in-order` ($\mathcal{R}_{io}$) on α `tree` to some notion of order captured by an abstract relation (ρ_{o}) on β

treefoldl

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treefoldl : α tree → β → (β → α → β) → β
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A parametric type can be constructed to relate `in-order` ($\mathcal{R}_{io}$) on α `tree` to some notion of order captured by an abstract relation (ρ_o) on β

$$(\rho_o)\ \texttt{treefoldl:}\ \{\texttt{t:}\alpha\ \texttt{tree}\} \longrightarrow \ldots$$

$$\longrightarrow \{\texttt{v:}\ \beta \mid \rho_o(\texttt{v}) = \mathcal{R}_{io}(\texttt{t})\}$$

A Parametric Type of `treefoldl`

$$(\rho_m,\rho_o)\ \mathtt{treefoldl}\colon \{t{:}\alpha\ \mathtt{tree}\} \longrightarrow \{b{:}\beta \mid \rho_m(b)=\emptyset$$

$$\wedge\ \rho_o(b)=\emptyset\}$$

$$\longrightarrow (\{xs{:}\beta\} \longrightarrow \{x{:}\alpha\} \longrightarrow$$

$$\{v{:}\beta \mid \rho_m(v)=\rho_m(xs)\ \cup\ \{x\}$$

$$\wedge\ \rho_o(v)=\rho_m(xs)\times\{x\}\ \cup\ \rho_o(xs)\})$$

$$\longrightarrow \{y{:}\ \beta \mid \rho_o(y)=R_{io}(t)\ \wedge\ \rho_m(y)=R_{tm}(t)\ \}$$

A Parametric Type of `treefoldl`

$$
\begin{aligned}
(\rho_m,\rho_o)\ \texttt{treefoldl}: \{t:\alpha\ \texttt{tree}\} \longrightarrow \{b:\beta \mid\ & \rho_m(b)=\varnothing \\
& \wedge\ \rho_o(b)=\varnothing\} \\
\longrightarrow \big(\{xs:\beta\} \longrightarrow \{x:\alpha\} \longrightarrow\ & \\
\{v:\beta \mid \rho_m(v) = \rho_m(xs)\ & \cup\ \{x\} \\
\wedge\ \rho_o(v) = \rho_m(xs)\times\{x\}\ & \cup\ \rho_o(xs)\}\big) \\
\longrightarrow \{y:\ \beta \mid \rho_o(y) = R_{io}(t)\ \wedge\ & \rho_m(y) = R_{tm}(t)\ \}
\end{aligned}
$$

Order invariant: relates `in-order` on the tree to a notion of order on β

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$$
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\{\texttt{v}:\beta \mid \rho_m(\texttt{v}) = \rho_m(\texttt{xs})\ &\cup\ \{\texttt{x}\} \\
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\longrightarrow \{\texttt{y}:\ \beta \mid \rho_o(\texttt{y}) = \mathcal{R}_{io}(\texttt{t})\ \wedge\ & \rho_m(\texttt{y}) = \mathcal{R}_{tm}(\texttt{t})\ \}
\end{aligned}
$$

Order invariant: relates `in-order` on the tree to a notion of order on β

Membership invariant: relates `membership` of the tree to a notion of membership of β

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$$
\begin{aligned}
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&\quad \longrightarrow \big(\{\mathtt{xs}:\beta\} \longrightarrow \{\mathtt{x}:\alpha\} \longrightarrow \\
&\qquad\qquad \{\mathtt{v}:\beta \mid \rho_m(\mathtt{v}) = \rho_m(\mathtt{xs})\ \cup\ \{\mathtt{x}\} \\
&\qquad\qquad\qquad \wedge\ \rho_o(\mathtt{v}) = \rho_m(\mathtt{xs})\times\{\mathtt{x}\}\ \cup\ \rho_o(\mathtt{xs})\}\big) \\
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\end{aligned}
$$

Order invariant: relates `in-order` on the tree to a notion of order on β

Membership invariant: relates `membership` of the tree to a notion of membership of β

inOrder using treefoldl

```
val inOrder = fn t => treefoldl ($R_{lm}$,$R_{fo}$) t []
                  (fn acc => fn x => acc ++ [x])
```

inOrder using treefoldl

$$\texttt{val inOrder = fn t => treefoldl}\ \overbrace{(R_{\texttt{lm}},R_{\texttt{fo}})}^{\text{Explicit relational parameter instantiation}}\ \texttt{t [ ]}$$

$$\texttt{(fn acc => fn x => acc ++ [x])}$$

inOrder using treefoldl

Explicit relational parameter instantiation

```
val inOrder = fn t => treefoldl (R_lm,R_fo) t []
            (fn acc => fn x => acc ++ [x])
```

has type

$$\{t{:}\alpha\ \texttt{tree}\} \longrightarrow \ldots \longrightarrow \{v{:}\ \alpha\ \texttt{list} \mid R_{fo}(v) = R_{io}(t) \wedge R_{lm}(v) = R_{tm}(t)\ \}$$

Parametric Relations

`id` and `pairMap` are functions parameterized over relations

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Relations can also be parameterized over relations

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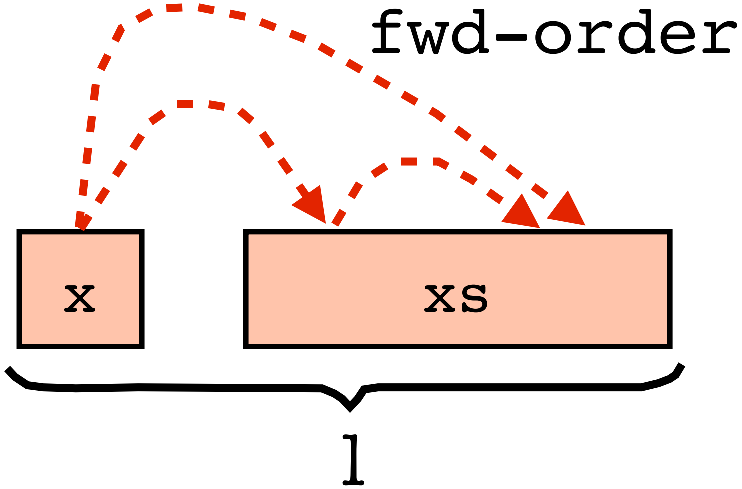
For Eg:

$$R_{fo}(l) = \{x\} \times R_{lm}(xs) \cup R_{fo}(xs)$$



Parametric Relations

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Relations can also be parameterized over relations

For Eg:

Relates elements of l

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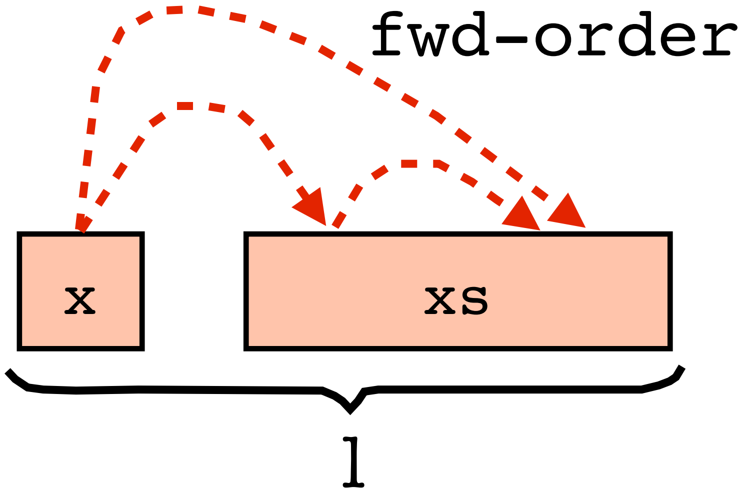


Parametric Relations

`id` and `pairMap` are functions parameterized over relations

Relations can also be parameterized over relations

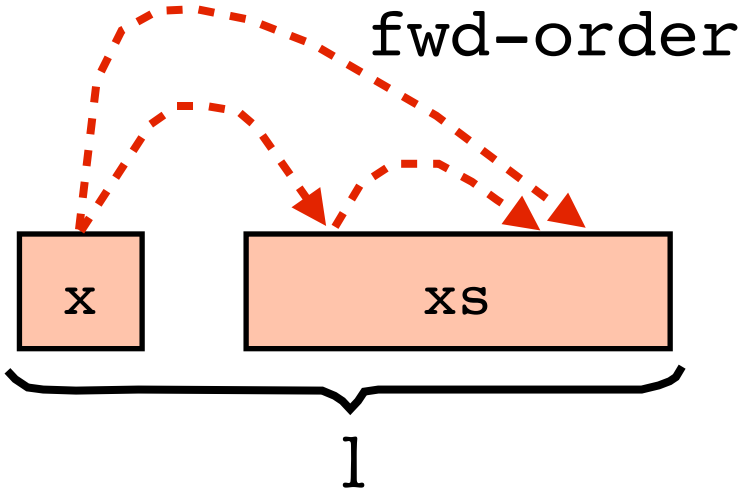
For Eg:

Relates elements of `l`

$R_{fo}(l) = \{x\} \times R_{lm}(xs) \cup R_{fo}(xs)$

Generalize

$R_{fo}[\rho](l) = \rho(x) \times R_{lm}[\rho](xs) \cup R_{fo}[\rho](xs)$



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Generalize

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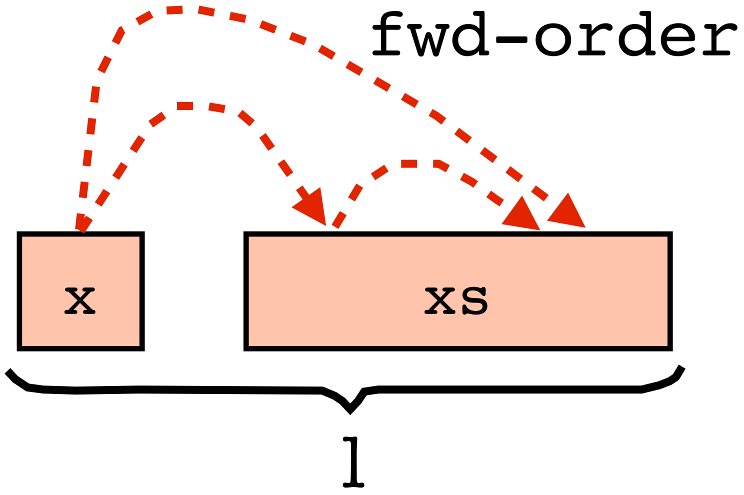
Relates different things for different instantiations of ρ



Parametric Relations

`id` and `pairMap` are functions parameterized over relations

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For Eg:

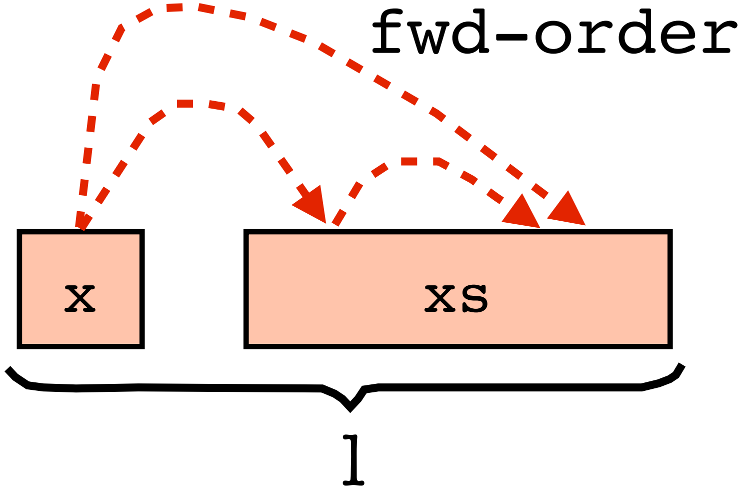
Relates elements of l

$$\mathrm{R}_{fo}(l) = \{x\} \times \mathrm{R}_{lm}(xs) \cup \mathrm{R}_{fo}(xs)$$

Generalize

$$\mathrm{R}_{fo}[\rho](l) = \rho(x) \times \mathrm{R}_{lm}[\rho](xs) \cup \mathrm{R}_{fo}[\rho](xs)$$

Relates different things for different instantiations of ρ



Note: If $\mathrm{R}_{id}(x)=\{x\}$ then $\mathrm{R}_{fo}[\mathrm{R}_{id}](l)$ relates elements like non-parametric $\mathrm{R}_{fo}(l)$

For Example …

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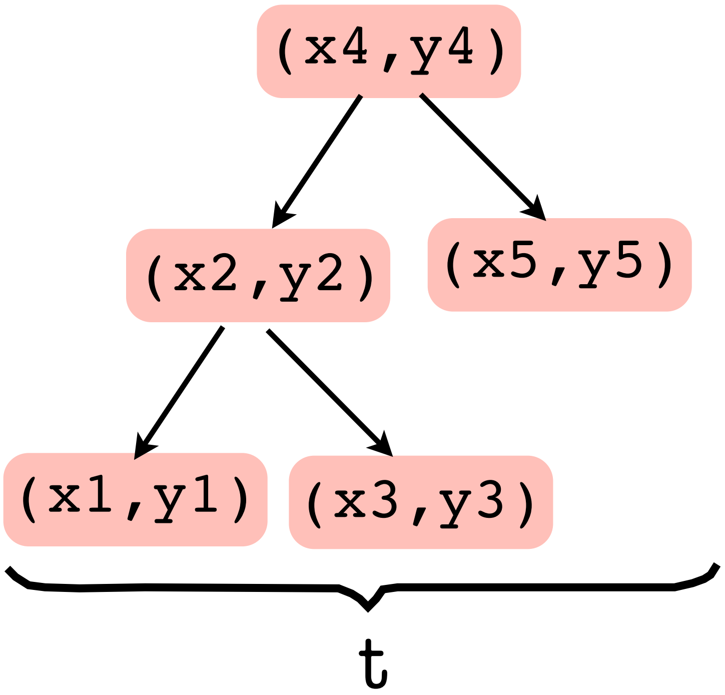
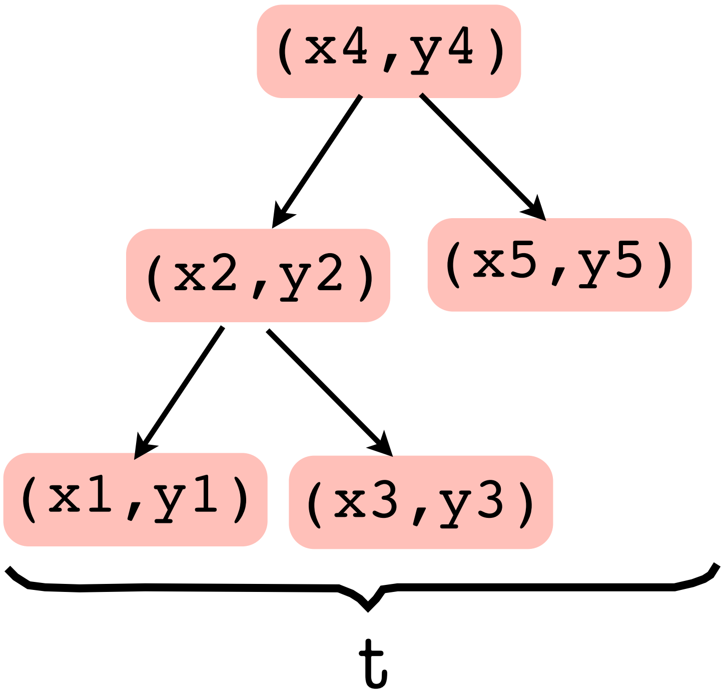


We know:

$$R_{io}(t) = \{((x_i, y_i), (x_j, y_j)) \mid i \leq j\}$$

For Example ...



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$$R_{io}(t) = \{((x_i, y_i), (x_j, y_j)) \mid i \leq j\}$$

By Definition:

$$R_{io}[\rho](t) = \{(\rho(x_i, y_i), \rho\,(x_j, y_j)) \mid i \leq j\}$$

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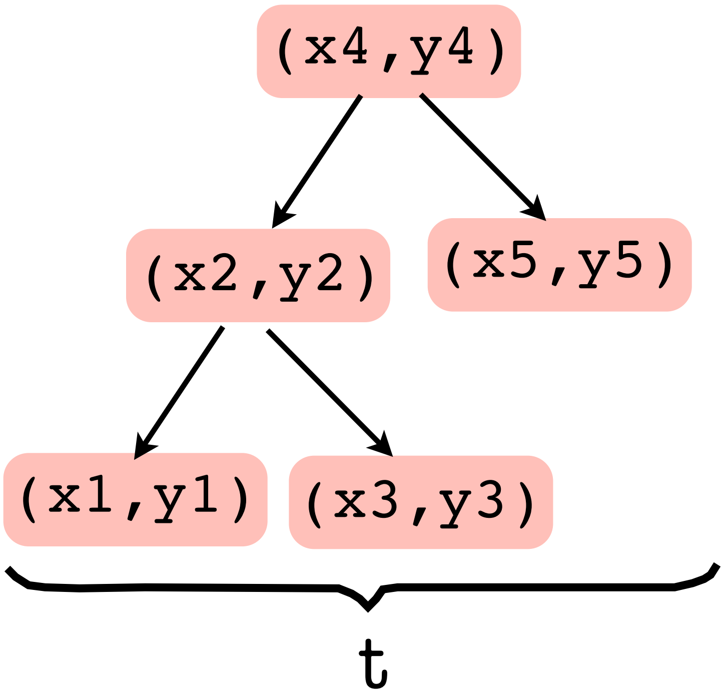

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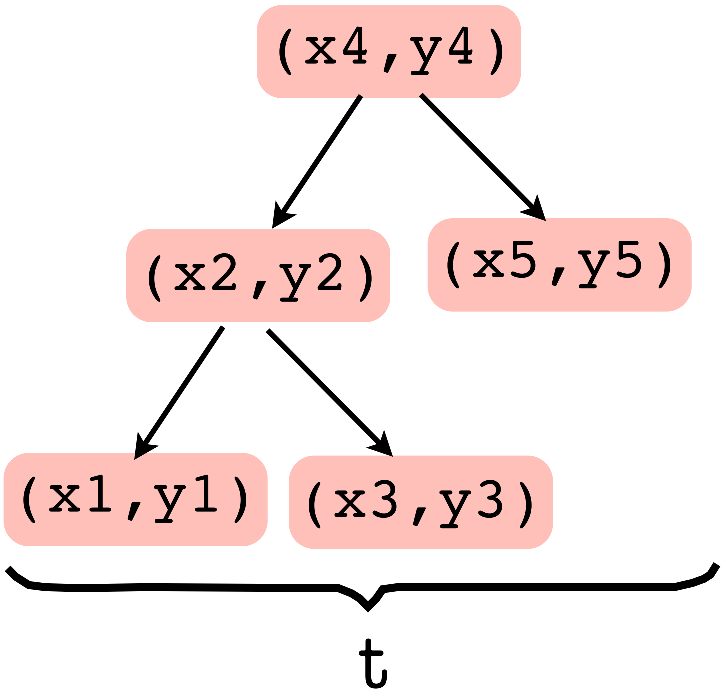
Let R_{fst} be a relation on pairs, such that

$$R_{fst}(x,y) = \{x\}$$

For Example ...



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Let R_{fst} be a relation on pairs, such that

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Now:

$$R_{io}[R_{fst}](t) = \{R_{fst}(x_i, y_i), R_{fst}\,(x_j, y_j)) \mid i \leq j\}$$

$$\Leftrightarrow \quad R_{io}[R_{fst}](t) = \{(x_i,x_j) \mid i \leq j\}$$

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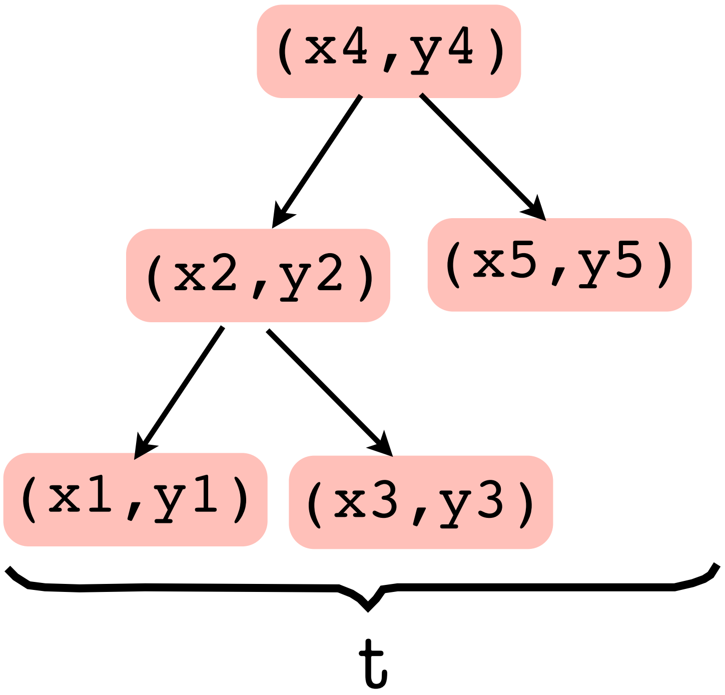

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in-order among first-components of pairs in t

For Example ...

For Example ...

$$\texttt{treeMap : } \alpha\ \texttt{tree} \longrightarrow (\alpha \longrightarrow \beta) \longrightarrow \beta\ \texttt{tree}$$

For Example ...

$$\texttt{treeMap} : \alpha\ \texttt{tree} \longrightarrow (\alpha \longrightarrow \beta) \longrightarrow \beta\ \texttt{tree}$$

Relational type ...

... by focusing on possible shape invariance between α and β (a la pairMap)

$$(\rho_\alpha,\rho_\beta)\ \texttt{treeMap} : \{t_1 : \alpha\ \texttt{tree}\}$$

$$\longrightarrow (\{x:\alpha\} \longrightarrow \{y:\beta \mid \rho_\beta(y) = \rho_\alpha(x)\})$$

For Example ...

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For Example ...

$$\texttt{treeMap} : \alpha\ \texttt{tree} \longrightarrow (\alpha \longrightarrow \beta) \longrightarrow \beta\ \texttt{tree}$$

$$\downarrow$$

Relational type ...

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$$(\rho_\alpha,\rho_\beta)\ \texttt{treeMap} : \{\texttt{t}_1 : \alpha\ \texttt{tree}\}$$

$$\longrightarrow (\{\texttt{x} : \alpha\} \longrightarrow \{\texttt{y} : \beta \mid \rho_\beta(\texttt{y}) = \rho_\alpha(\texttt{x})\})$$

$$\longrightarrow \{\texttt{t}_2 : \beta\ \texttt{tree} \mid ?\ \}$$

For Example ...

$$\texttt{treeMap} : \alpha\ \texttt{tree} \longrightarrow (\alpha \longrightarrow \beta) \longrightarrow \beta\ \texttt{tree}$$

Relational type ... $\Big\downarrow$ ... by focusing on possible shape invariance between α and β (a la `pairMap`)

$$(\rho_\alpha, \rho_\beta)\ \texttt{treeMap} : \{t_1 : \alpha\ \texttt{tree}\}$$
$$\longrightarrow (\{x:\alpha\} \longrightarrow \{y:\beta \mid \rho_\beta(y) = \rho_\alpha(x)\})$$
$$\longrightarrow \{t_2:\beta\ \texttt{tree} \mid ?\ \}$$

$$R_{io}(t_2) \cancel{=} R_{io}(t_1)$$

$R_{io}(t_i)$ is a relation on elements of t_i

and elements of $t_1 \neq$ elements of t_2

For Example ...

$$\texttt{treeMap} : \alpha\ \texttt{tree} \longrightarrow (\alpha \longrightarrow \beta) \longrightarrow \beta\ \texttt{tree}$$

Relational type ...

... by focusing on possible shape invariance between α and β (a la `pairMap`)

$$\begin{aligned}
(\rho_\alpha,\rho_\beta)\ \texttt{treeMap} :\ & \{t_1 : \alpha\ \texttt{tree}\} \\
& \longrightarrow (\{x:\alpha\} \longrightarrow \{y:\beta \mid \rho_\beta(y) = \rho_\alpha(x)\}) \\
& \longrightarrow \{t_2:\beta\ \texttt{tree} \mid \mathcal{R}_{io}[\rho_\beta](t_2) = \mathcal{R}_{io}[\rho_\alpha](t_1)\}
\end{aligned}$$

For Example ...

$$\texttt{treeMap} : \alpha\ \texttt{tree} \longrightarrow (\alpha \longrightarrow \beta) \longrightarrow \beta\ \texttt{tree}$$

Relational type ... ↓ ... by focusing on possible shape invariance between α and β (a la $\texttt{pairMap}$)

$$\begin{aligned}(\rho_\alpha,\rho_\beta)\ \texttt{treeMap} : &\ \{t_1 : \alpha\ \texttt{tree}\}\\ &\longrightarrow (\{x:\alpha\} \longrightarrow \{y:\beta \mid \rho_\beta(y) = \rho_\alpha(x)\})\\ &\longrightarrow \{t_2:\beta\ \texttt{tree} \mid \mathcal{R}_{io}[\rho_\beta](t_2) = \mathcal{R}_{io}[\rho_\alpha](t_1)\}\end{aligned}$$

Parametric in-order relation ($\mathcal{R}_{io}[\rho]$) is not necessarily a relation over elements.

For Example ...

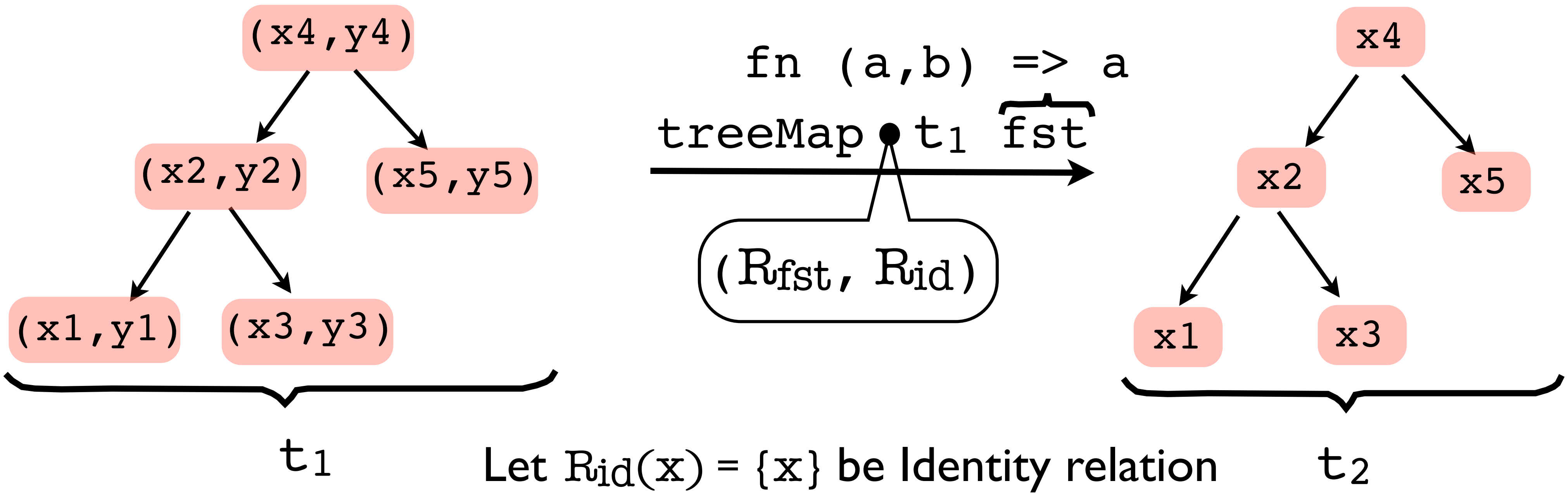
$$
\begin{aligned}
&(\rho_\alpha,\rho_\beta)\ \texttt{treeMap} : \{\texttt{t}_1{:}\alpha\ \texttt{tree}\}\\
&\qquad\qquad \longrightarrow \left(\{\texttt{x}{:}\alpha\} \longrightarrow \{\texttt{y}{:}\beta \mid \rho_\beta(\texttt{y}) = \rho_\alpha(\texttt{x})\}\right)\\
&\qquad\qquad\quad \longrightarrow \{\texttt{t}_2{:}\beta\ \texttt{tree} \mid \mathcal{R}_{\mathrm{io}}[\rho_\beta](\texttt{t}_2) = \mathcal{R}_{\mathrm{io}}[\rho_\alpha](\texttt{t}_1)\}
\end{aligned}
$$

For Example ...

$$
\begin{aligned}
&(\rho_\alpha,\rho_\beta)\ \mathtt{treeMap} : \{t_1{:}\alpha\ \mathtt{tree}\}\\
&\qquad\longrightarrow \big(\{x{:}\alpha\} \longrightarrow \{y{:}\beta \mid \rho_\beta(y) = \rho_\alpha(x)\}\big)\\
&\qquad\quad\longrightarrow \{t_2{:}\beta\ \mathtt{tree} \mid R_{io}[\rho_\beta](t_2) = R_{io}[\rho_\alpha](t_1)\}
\end{aligned}
$$

For Example ...

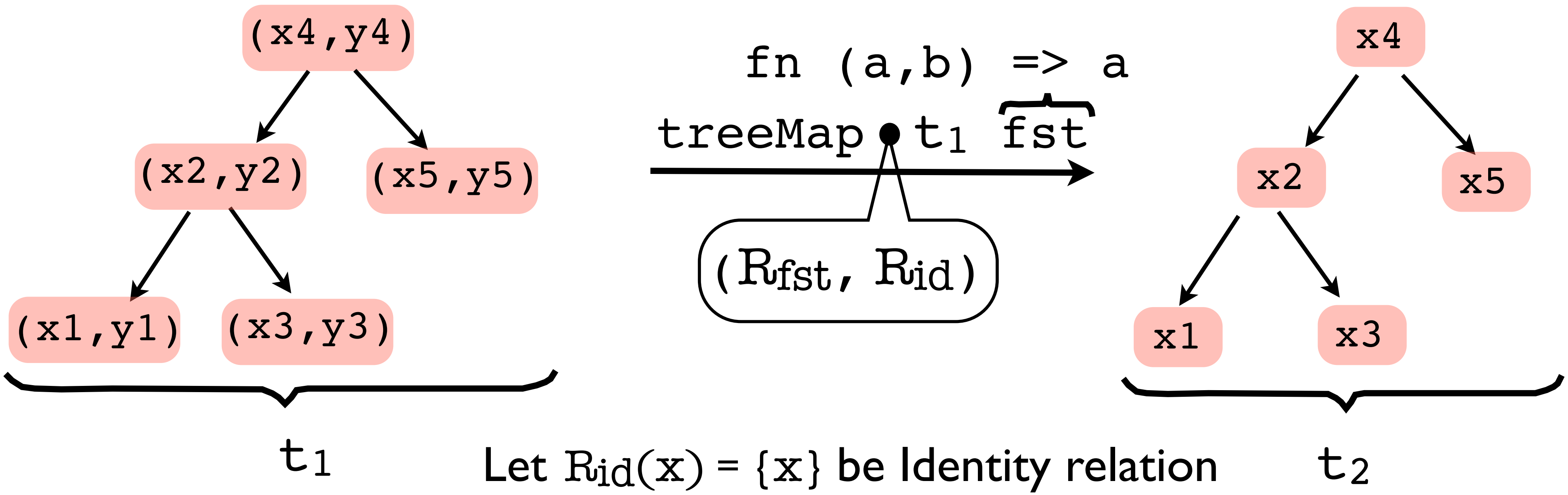
$$\texttt{treeMap}\ (\mathcal{R}_{fst},\ \mathcal{R}_{id}) : \{t_1 : \alpha\ \texttt{tree}\}$$
$$\longrightarrow (\{x:\alpha\} \longrightarrow \{y:\beta \mid \mathcal{R}_{id}(y) = \mathcal{R}_{fst}(x)\})$$
$$\longrightarrow \{t_2:\beta\ \texttt{tree} \mid \mathcal{R}_{io}[\mathcal{R}_{id}](t_2) = \mathcal{R}_{io}[\mathcal{R}_{fst}](t_1)\}$$



Let $\mathcal{R}_{id}(x) = \{x\}$ be Identity relation

For Example ...

$$\begin{aligned}&\texttt{treeMap}\ (\mathcal{R}_{fst},\ \mathcal{R}_{id}) : \{t_1 : \alpha\ \texttt{tree}\}\\ &\quad \longrightarrow (\{x:\alpha\} \longrightarrow \{y:\beta \mid \mathcal{R}_{id}(y) = \mathcal{R}_{fst}(x)\})\\ &\qquad \longrightarrow \{t_2:\beta\ \texttt{tree} \mid \mathcal{R}_{io}[\mathcal{R}_{id}](t_2) = \mathcal{R}_{io}[\mathcal{R}_{fst}](t_1)\}\end{aligned}$$



Let $\mathcal{R}_{id}(x) = \{x\}$ be Identity relation

in-order among elements of t_2 = in-order among first components of pairs in t_1

So far ...

So far ...

- Relational language to express shapes

So far ...

- Relational language to express shapes
- Functions parameterized on relations

So far ...

- Relational language to express shapes
- Functions parameterized on relations
- Relations parameterized on relations

So far ...

- Relational language to express shapes
- Functions parameterized on relations
- Relations parameterized on relations

Expressive type language

So far ...

- Relational language to express shapes
- Functions parameterized on relations
- Relations parameterized on relations

Expressive type language

For type-based shape analysis to be effective, we need type checking with such expressive types to be decidable and practical

Decidability

Type checking is decidable if type refinements can be encoded in a decidable logic

$$\frac{\Gamma \vdash \{\nu : T \mid \phi_1\} \qquad \Gamma \vdash \{\nu : T \mid \phi_2\} \qquad [\![\Gamma_R]\!] \models [\![\Gamma, \nu : T]\!] \Rightarrow [\![\phi_1]\!] \Rightarrow [\![\phi_2]\!]}{\Gamma \vdash \{\nu : T \mid \phi_1\} <: \{\nu : T \mid \phi_2\}}$$

i.e., if ϕ is a type refinement, then $[\![\phi]\!]$ must be an expression in a decidable logic

For the language of relational type refinements,
there exists such an encoding into a decidable
subset of many-sorted first-order logic (MSFOL)

$\Rightarrow$

Type checking is decidable

MSFOL

Many-sorted first-order logic is a syntactic extension of first-order logic with sorts (types)

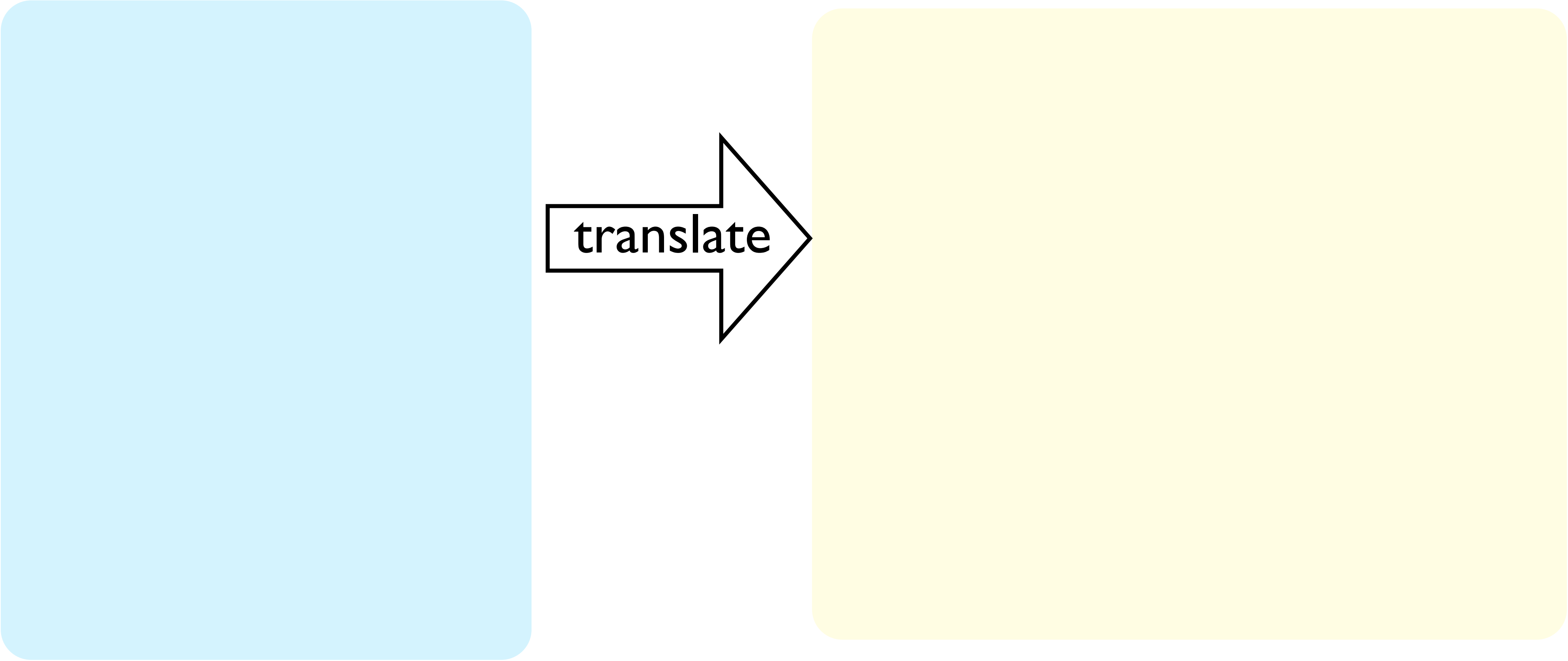
We consider a decidable subset with ...

Effectively Propositional (EPR) MSFOL

Uninterpreted sorts	$\mathtt{T_0,\ T_1,\ \ldots}$
Sorted variables	$\mathtt{x{:}T_0,\ y{:}T_1,\ \ldots}$
Sorted uninterpreted boolean functions (relations)	$\mathtt{R{:}T_0 \longrightarrow bool\ \ \ldots}$
Prenex quantification over sorted variables	$\mathtt{\forall(k{:}T_0).R(x,k) \Leftrightarrow x{=}k,}$ $\mathtt{\exists(j{:}T_0).f(y) = j}$

Encoding ...

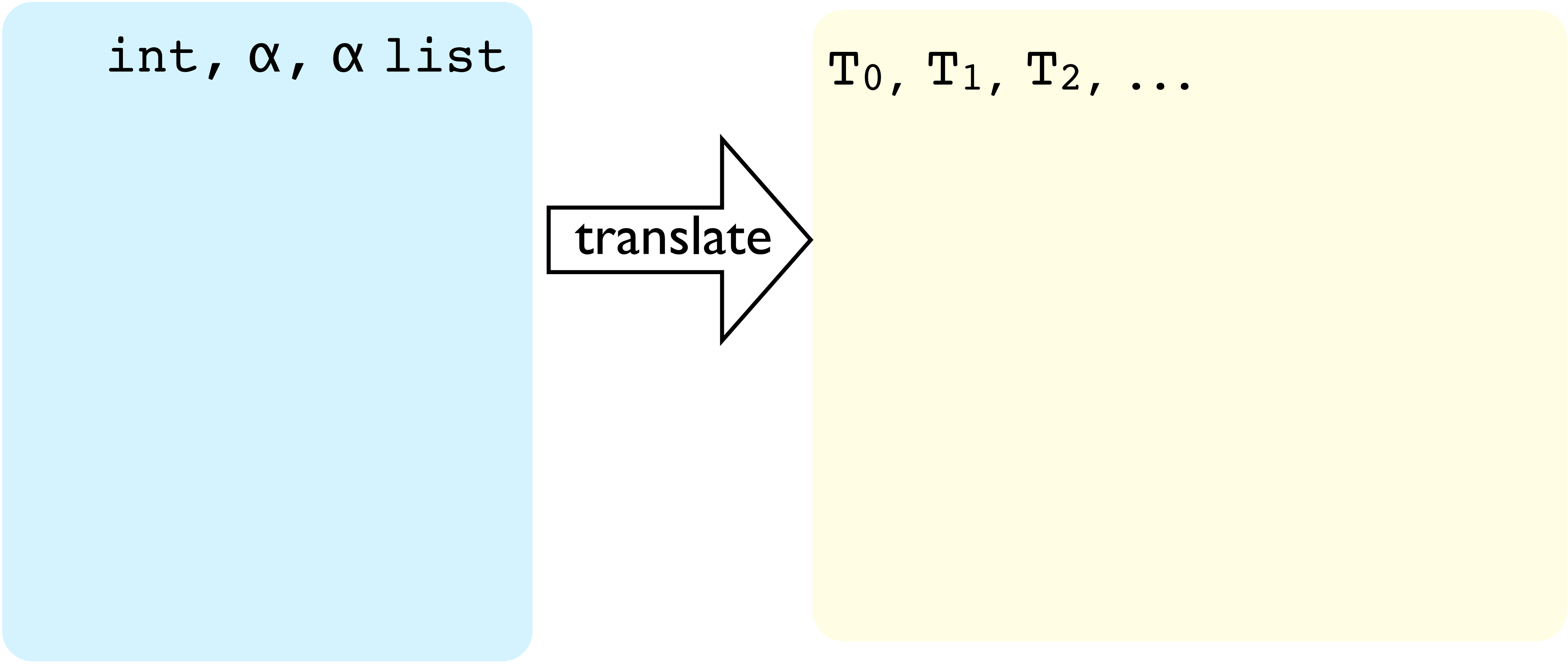
... is translation of artifacts of type refinement language into the EPR fragment of MSFOL.



translate

Encoding ...

... is translation of artifacts of type refinement language into the EPR fragment of MSFOL.



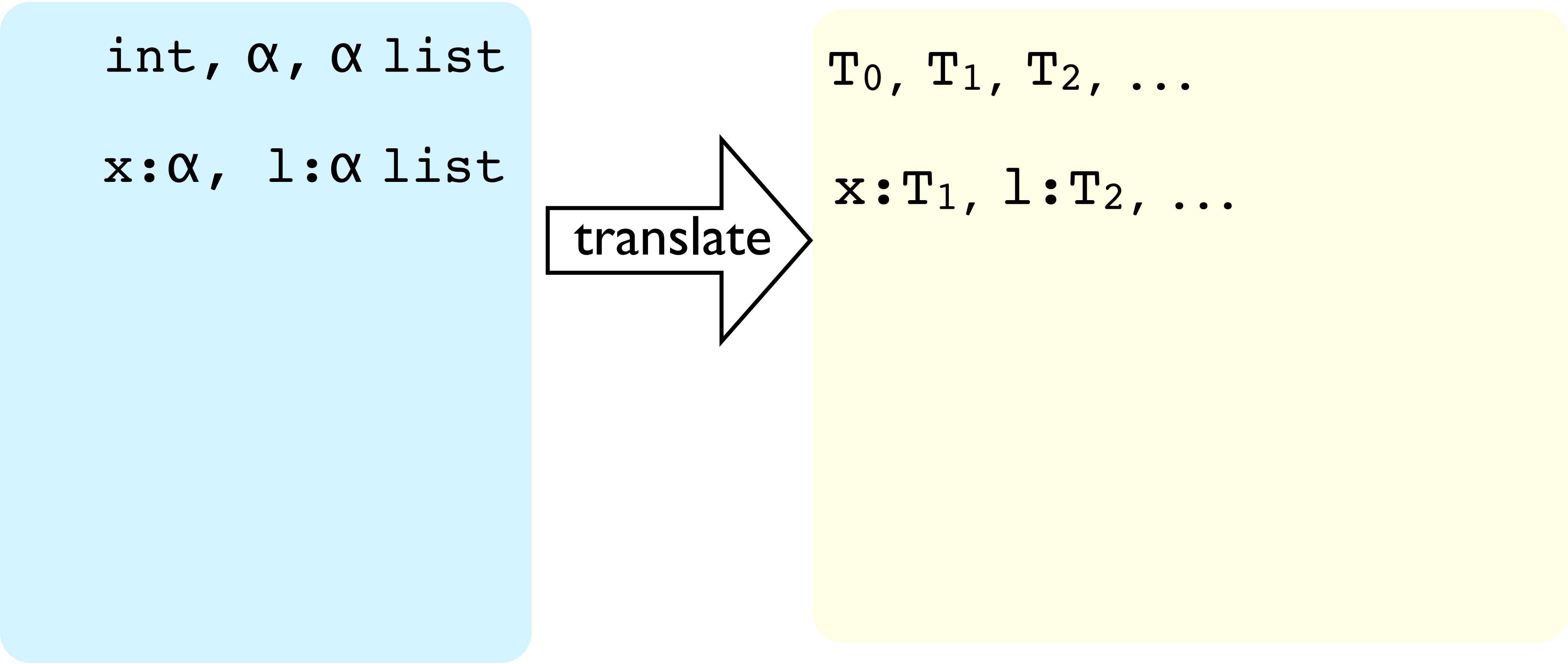
`int, α, α list`

translate

T_0, T_1, T_2, ...

Encoding ...

... is translation of artifacts of type refinement language into the EPR fragment of MSFOL.



`int`, α, α `list`

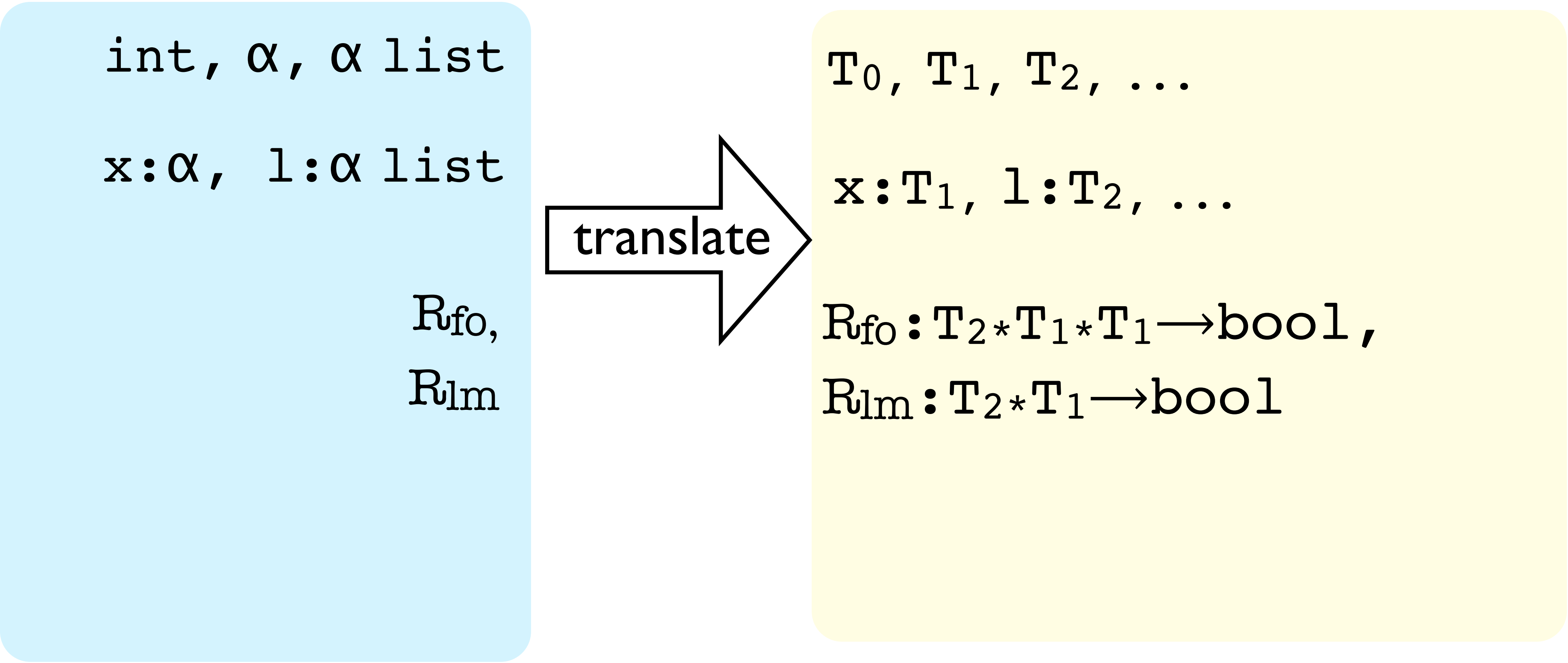
`x`:α, `l`:α `list`

$\mathtt{T}_0$, $\mathtt{T}_1$, $\mathtt{T}_2$, ...

$\mathtt{x}$:$\mathtt{T}_1$, $\mathtt{l}$:$\mathtt{T}_2$, ...

Encoding ...

... is translation of artifacts of type refinement language into the EPR fragment of MSFOL.



`int, α, α list`

`x:α, l:α list`

R_{fo},

R_{lm}

translate

`T0, T1, T2, ...`

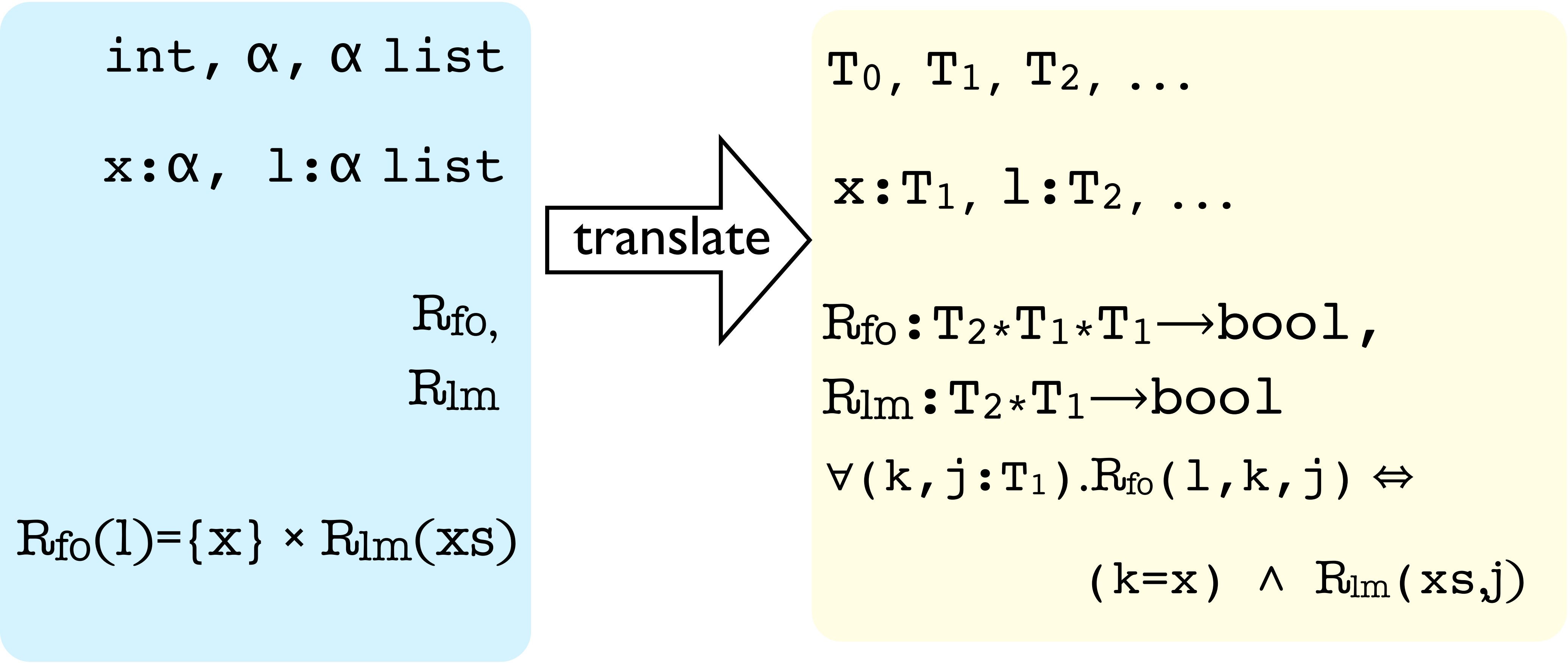
`x:T1, l:T2, ...`

R_{fo} `:T2*T1*T1`$\longrightarrow$`bool,`

R_{lm} `:T2*T1`$\longrightarrow$`bool`

Encoding ...

... is translation of artifacts of type refinement language into the EPR fragment of MSFOL.



$\texttt{int, α, α list}$

$\texttt{x:α, l:α list}$

$\mathcal{R}_{fo}$,

$\mathcal{R}_{lm}$

$\mathcal{R}_{fo}(l)=\{x\} \times \mathcal{R}_{lm}(xs)$

translate

$T_0, T_1, T_2, \ldots$

$x{:}T_1,\ l{:}T_2, \ldots$

$\mathcal{R}_{fo} : T_2 * T_1 * T_1 \longrightarrow \texttt{bool},$

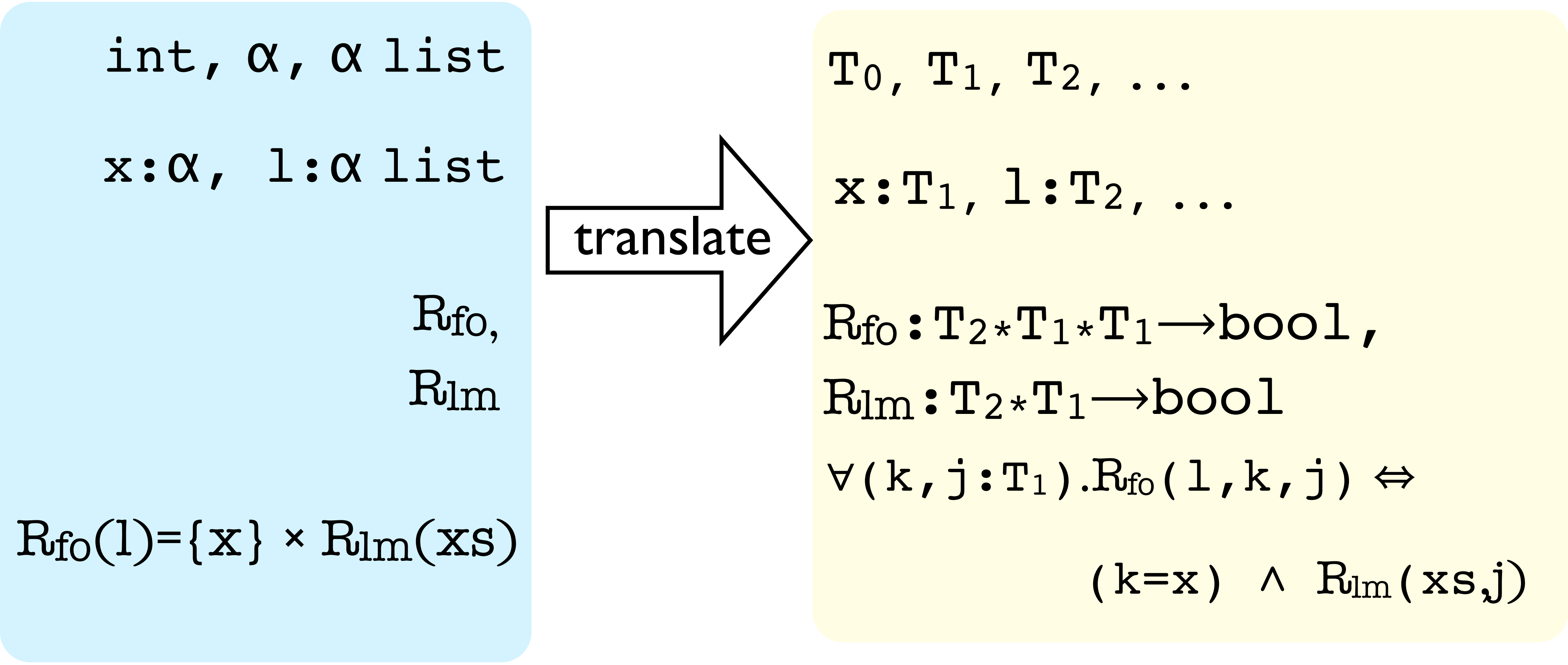
$\mathcal{R}_{lm} : T_2 * T_1 \longrightarrow \texttt{bool}$

$\forall(k,j{:}T_1).\mathcal{R}_{fo}(l,k,j) \Leftrightarrow$

$(k=x)\ \wedge\ \mathcal{R}_{lm}(xs,j)$

Encoding ...

... is translation of artifacts of type refinement language into the EPR fragment of MSFOL.



int, α, α list

x:α, l:α list

$\mathcal{R}_{fo}$,

$\mathcal{R}_{lm}$

$\mathcal{R}_{fo}(l)=\{x\} \times \mathcal{R}_{lm}(xs)$

translate

T_0, T_1, T_2, ...

x:T_1, l:T_2, ...

$\mathcal{R}_{fo} : T_2 * T_1 * T_1 \longrightarrow$ bool,

$\mathcal{R}_{lm} : T_2 * T_1 \longrightarrow$ bool

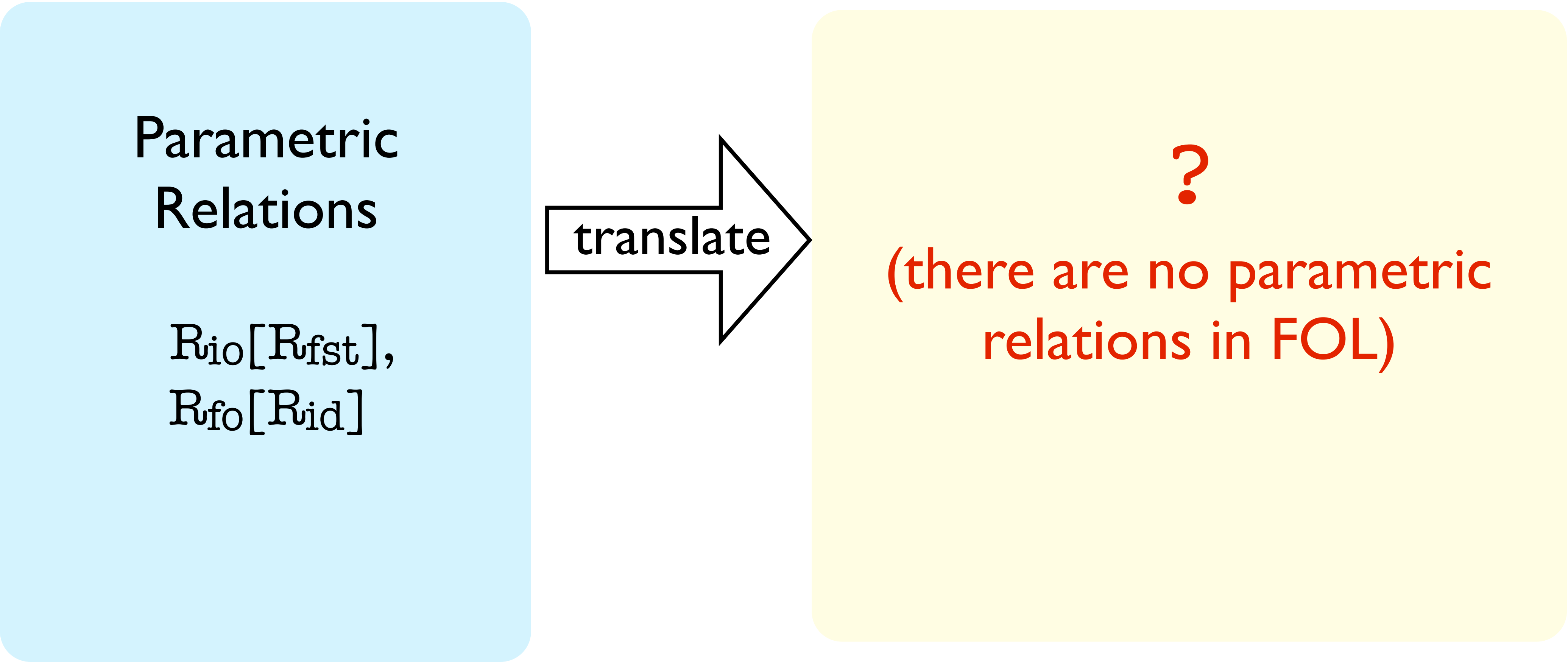
$\forall (k, j : T_1).\mathcal{R}_{fo}(l, k, j) \Leftrightarrow$

$(k=x) \wedge \mathcal{R}_{lm}(xs, j)$

but ...

Encoding ...

... parametric relations is not straightforward



Parametric Relations

$R_{io}[R_{fst}]$, $R_{fo}[R_{id}]$

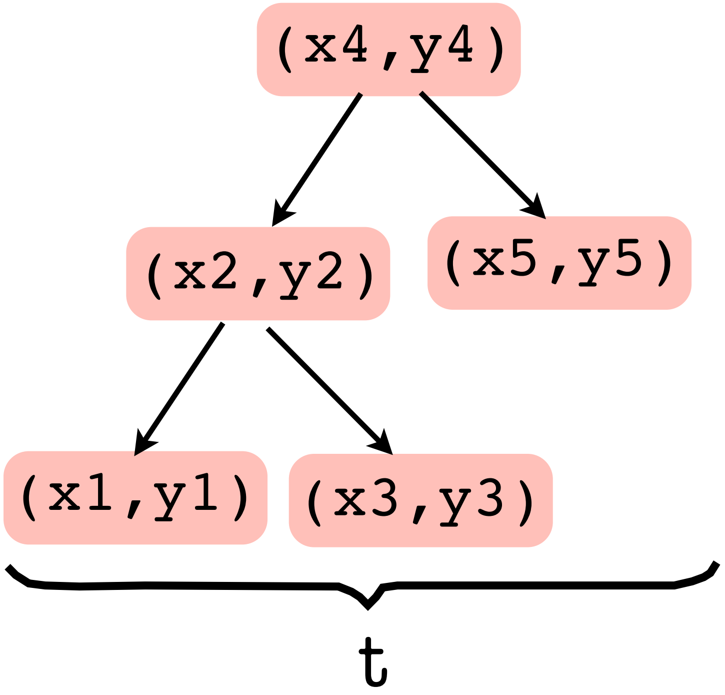
translate

?

(there are no parametric relations in FOL)

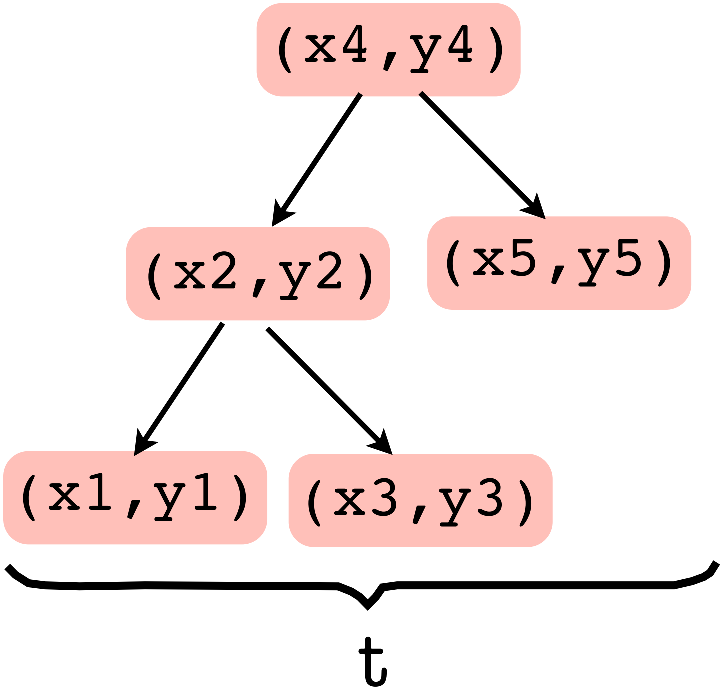
A fully instantiated parametric relation can be defined in terms of its component non-parametric relations

For eg:



A fully instantiated parametric relation can be defined in terms of its component non-parametric relations

For eg:



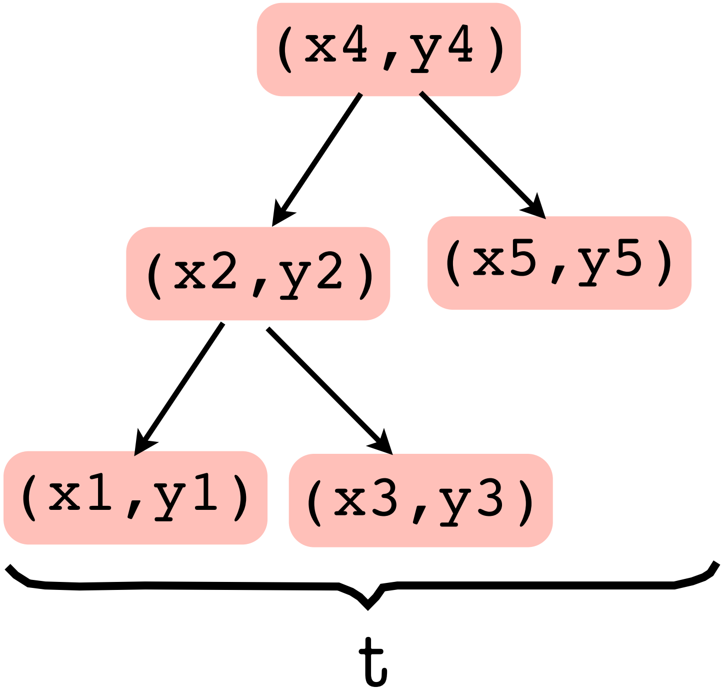
We have already seen:

$$R_{io}(t) = \{((x_i, y_i), (x_j, y_j)) \mid i \leq j\}$$

$$R_{io}[R_{fst}](t) = \{(x_i, x_j) \mid i \leq j\}$$

A fully instantiated parametric relation can be defined in terms of its component non-parametric relations

For eg:



We have already seen:

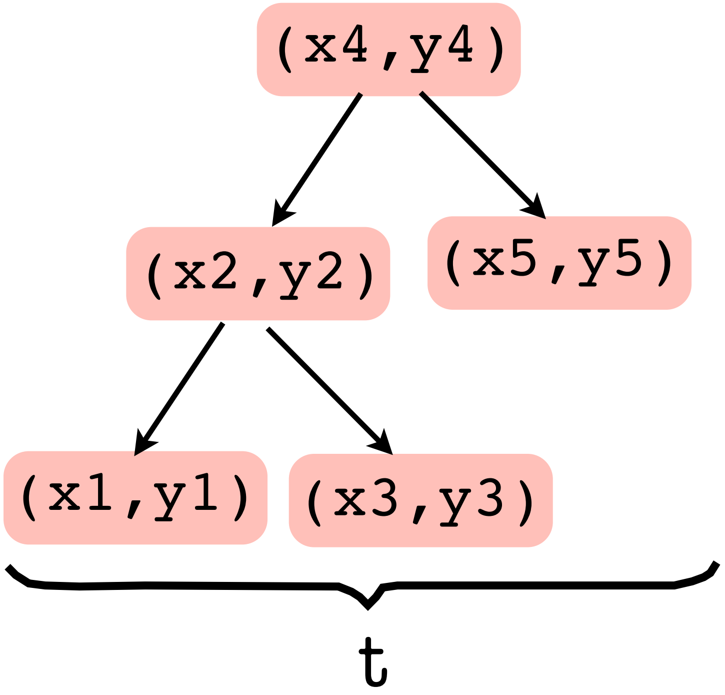
$$R_{io}(t) = \{((x_i, y_i), (x_j, y_j)) \mid i \leq j\}$$

R_{fst}

$$R_{io}[R_{fst}](t) = \{(x_i, x_j) \mid i \leq j\}$$

A fully instantiated parametric relation can be defined in terms of its component non-parametric relations

For eg:



We have already seen:

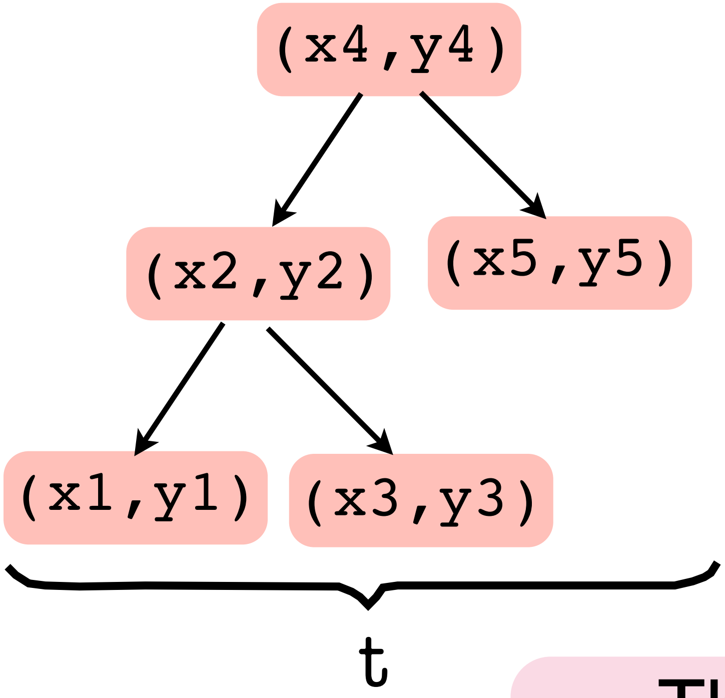
$$R_{io}(t) = \{((x_i, y_i), (x_j, y_j)) \mid i \leq j\}$$

$$\begin{tikzcd} (x_i, y_i) \arrow[d, "R_{fst}"'] & (x_j, y_j) \arrow[d, "R_{fst}"] \\ x_i & x_j \end{tikzcd}$$

$$R_{io}[R_{fst}](t) = \{(x_i, x_j) \mid i \leq j\}$$

A fully instantiated parametric relation can be defined in terms of its component non-parametric relations

For eg:



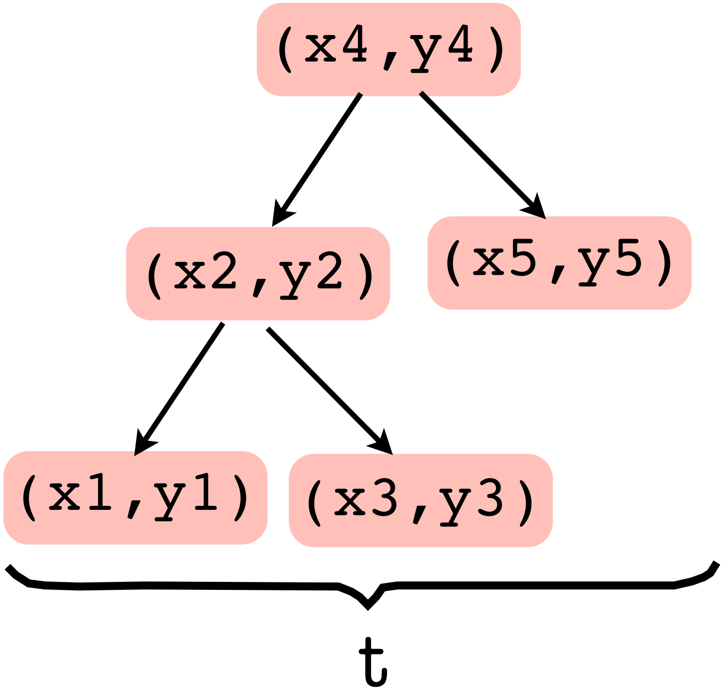
We have already seen:

$$\begin{tikzcd} R_{io}(t) = \{(& (x_i, y_i), \arrow[d, "R_{fst}"'] & (x_j, y_j) \arrow[d, "R_{fst}"] &) \mid i \leq j\} \\ R_{io}[R_{fst}](t) = \{(& x_i, & x_j &) \mid i \leq j\} \end{tikzcd}$$

The set $R_{io}[R_{fst}](t)$ is obtained from the set $R_{io}(t)$ by mapping both components of pairs with R_{fst}

A fully instantiated parametric relation can be defined in terms of its component non-parametric relations

For eg:



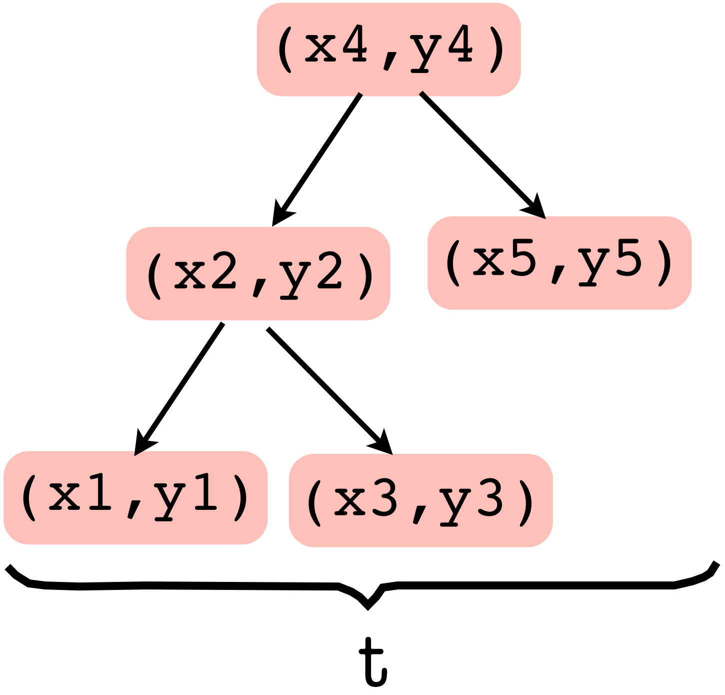
We have already seen:

$$R_{io}(t) = \{((x_i, y_i), (x_j, y_j)) \mid i \leq j\}$$

$$R_{io}[R_{fst}](t) = \{(x_i, x_j) \mid i \leq j\}$$

A fully instantiated parametric relation can be defined in terms of its component non-parametric relations

For eg:



We have already seen:

$$R_{io}(t) = \{((x_i, y_i), (x_j, y_j)) \mid i \leq j\}$$

$$R_{io}[R_{fst}](t) = \{(x_i, x_j) \mid i \leq j\}$$

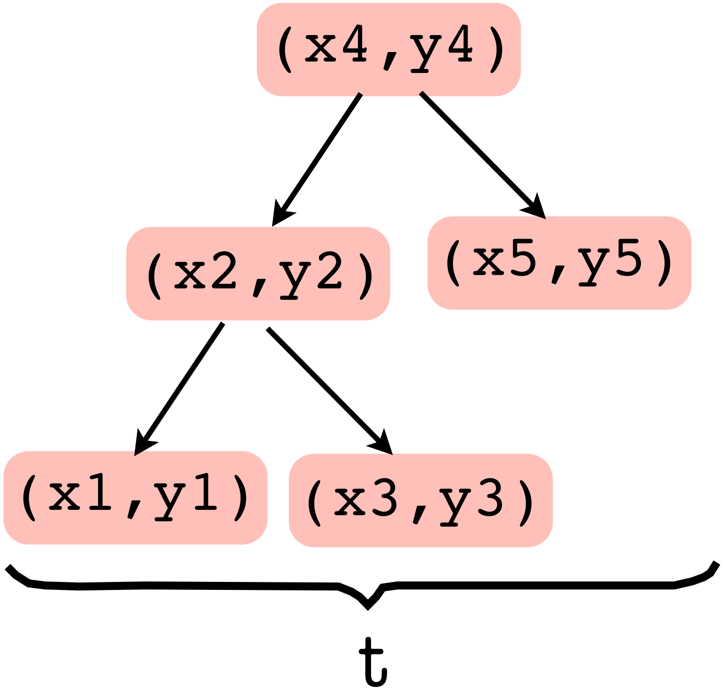
$$\Leftrightarrow$$

$$R_{io}[R_{fst}](t) =$$

$$\{(R_{fst}(a), R_{fst}(b)) \mid (a,b) \in R_{io}(t)\}$$

A fully instantiated parametric relation can be defined in terms of its component non-parametric relations

For eg:



We have already seen:

$$R_{io}(t) = \{((x_i, y_i), (x_j, y_j)) \mid i \leq j\}$$

$$R_{io}[R_{fst}](t) = \{(x_i, x_j) \mid i \leq j\}$$

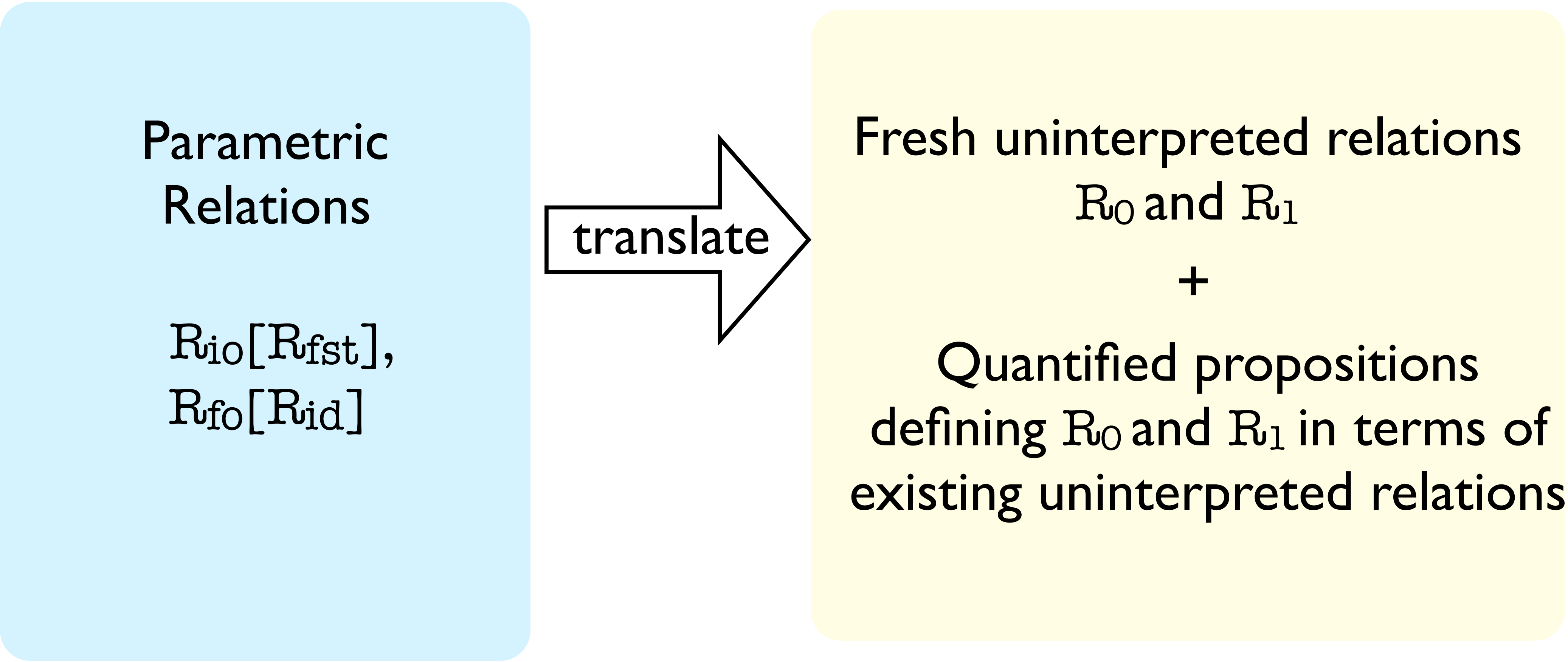
$$\Leftrightarrow$$

$$R_{io}[R_{fst}](t) = \{(R_{fst}(a), R_{fst}(b)) \mid (a,b) \in R_{io}(t)\}$$

Defines $R_{io}[R_{fst}]$ in terms of R_{io} and R_{fst}

Encoding ...

... parametric relations by defining them in terms of their component non-parametric relations



Parametric Relations

$R_{io}[R_{fst}]$, $R_{fo}[R_{id}]$

translate

Fresh uninterpreted relations R_0 and R_1

+

Quantified propositions defining R_0 and R_1 in terms of existing uninterpreted relations

Off-the-shelf SMT solvers (eg: Z3) are efficient decision procedures for the EPR fragment of MSFOL.

Off-the-shelf SMT solvers (eg: Z3) are efficient decision procedures for the EPR fragment of MSFOL.

⇒

A practical type checker can be constructed by encoding type refinements in MSFOL and using SMT solvers for subtype checking.

Off-the-shelf SMT solvers (eg: Z3) are efficient decision procedures for the EPR fragment of MSFOL.

CATALYST ⇒

A $\overbrace{\text{practical type checker}}$ can be constructed by encoding type refinements in MSFOL and using SMT solvers for subtype checking.

CATALYST

Implemented as extended type checking pass in MLton Standard ML compiler

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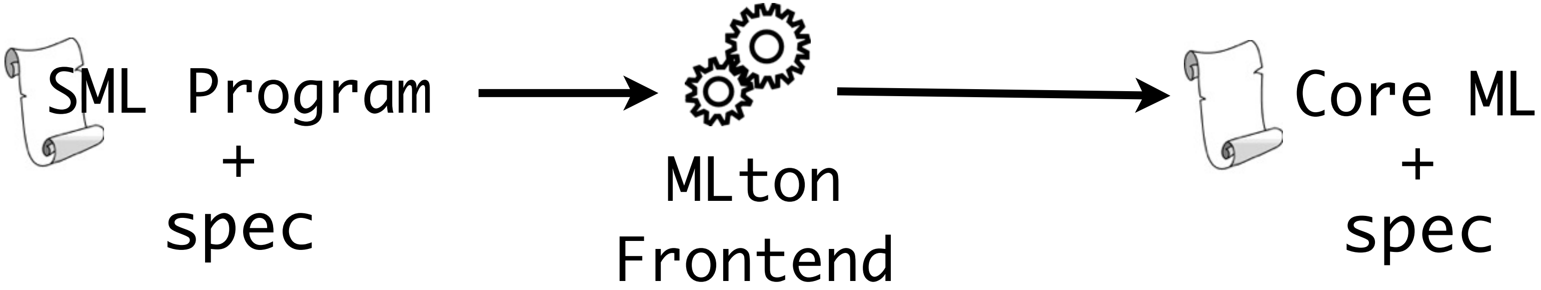
Implemented as extended type checking pass in MLton Standard ML compiler

SML Program

+

spec

Implemented as extended type checking pass in MLton Standard ML compiler



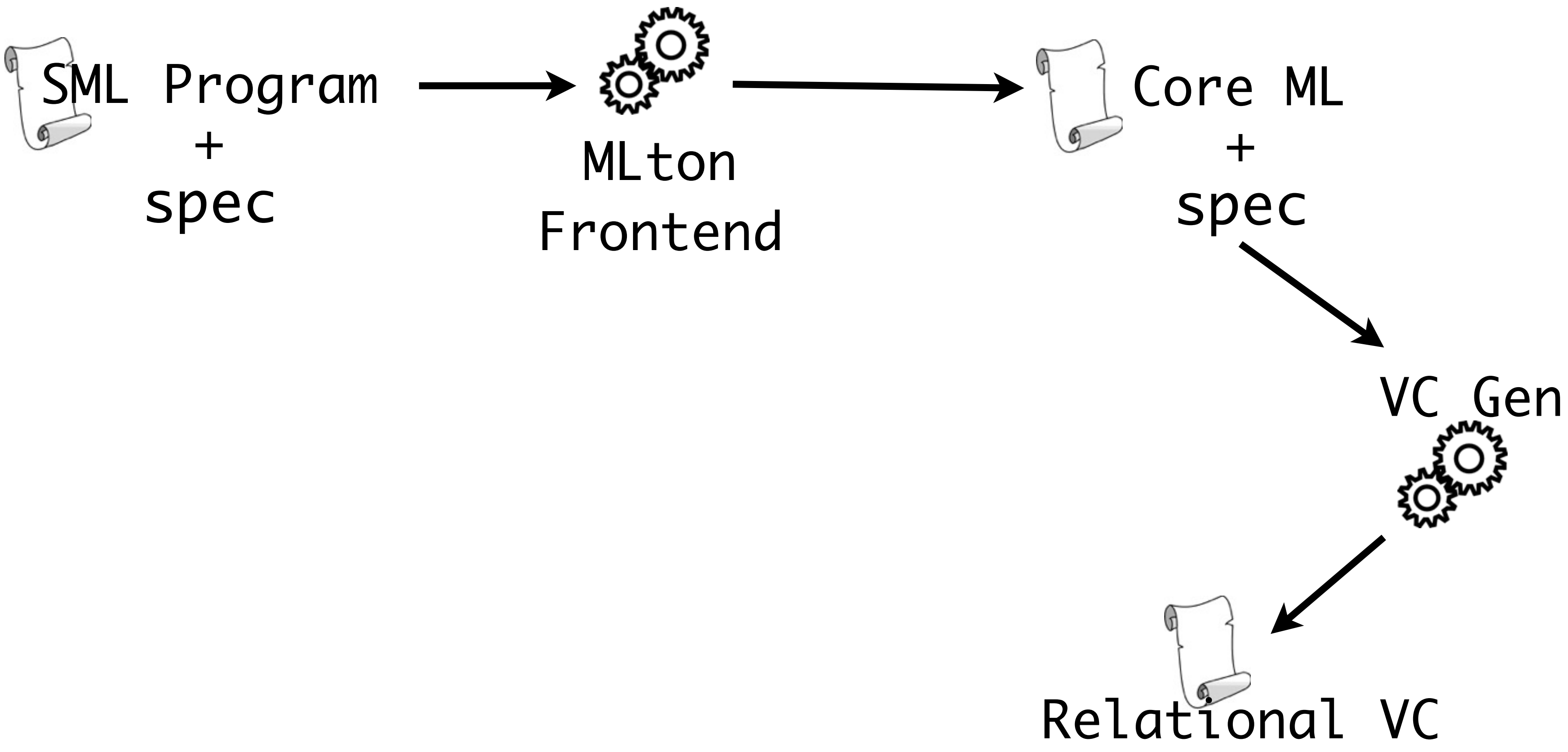
SML Program
+
spec

MLton
Frontend

Core ML
+
spec

CATALYST

Implemented as extended type checking pass in MLton Standard ML compiler



SML Program
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VC Gen

Relational VC

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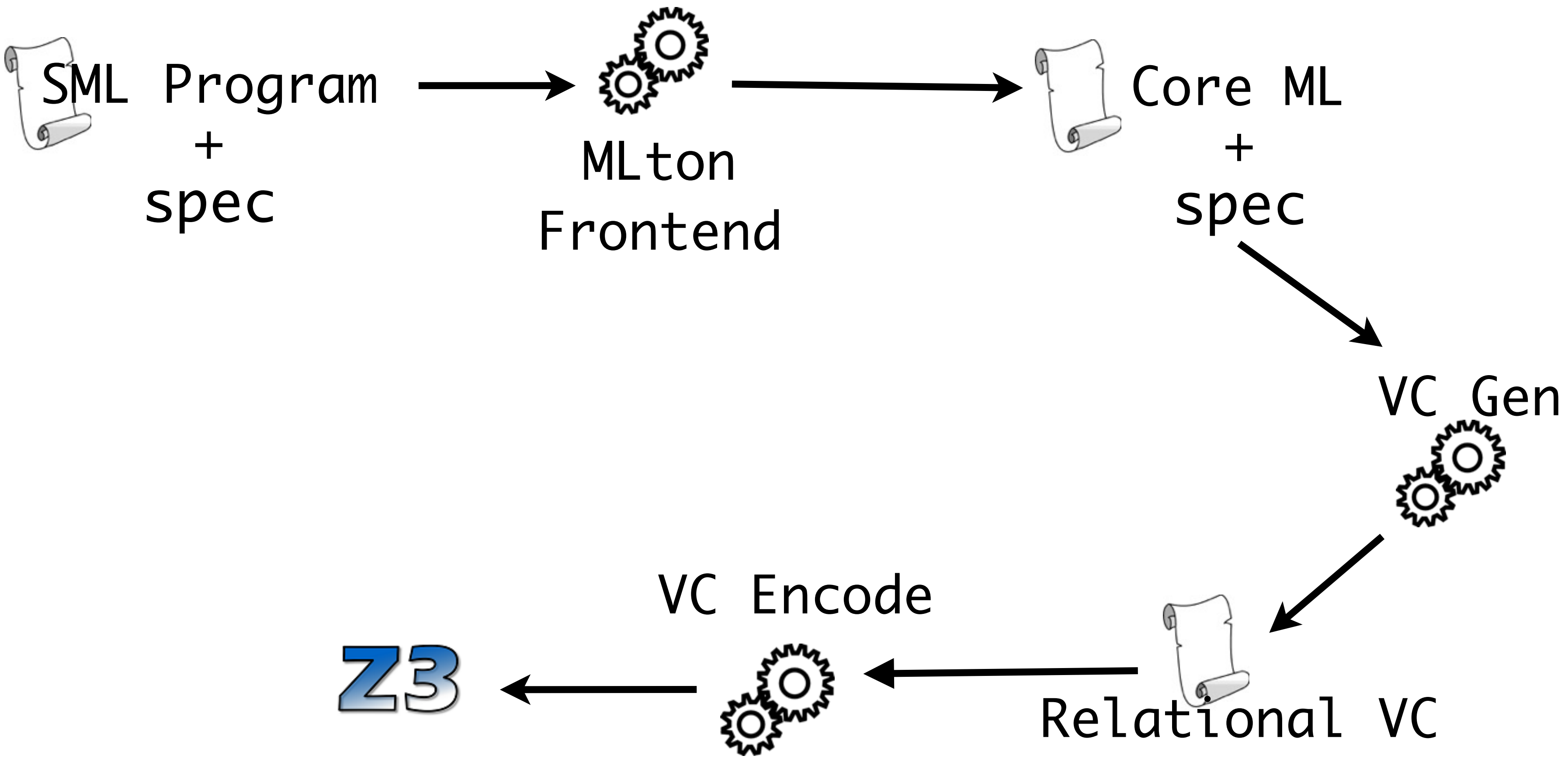

SML Program + spec

MLton Frontend

Core ML + spec

VC Gen

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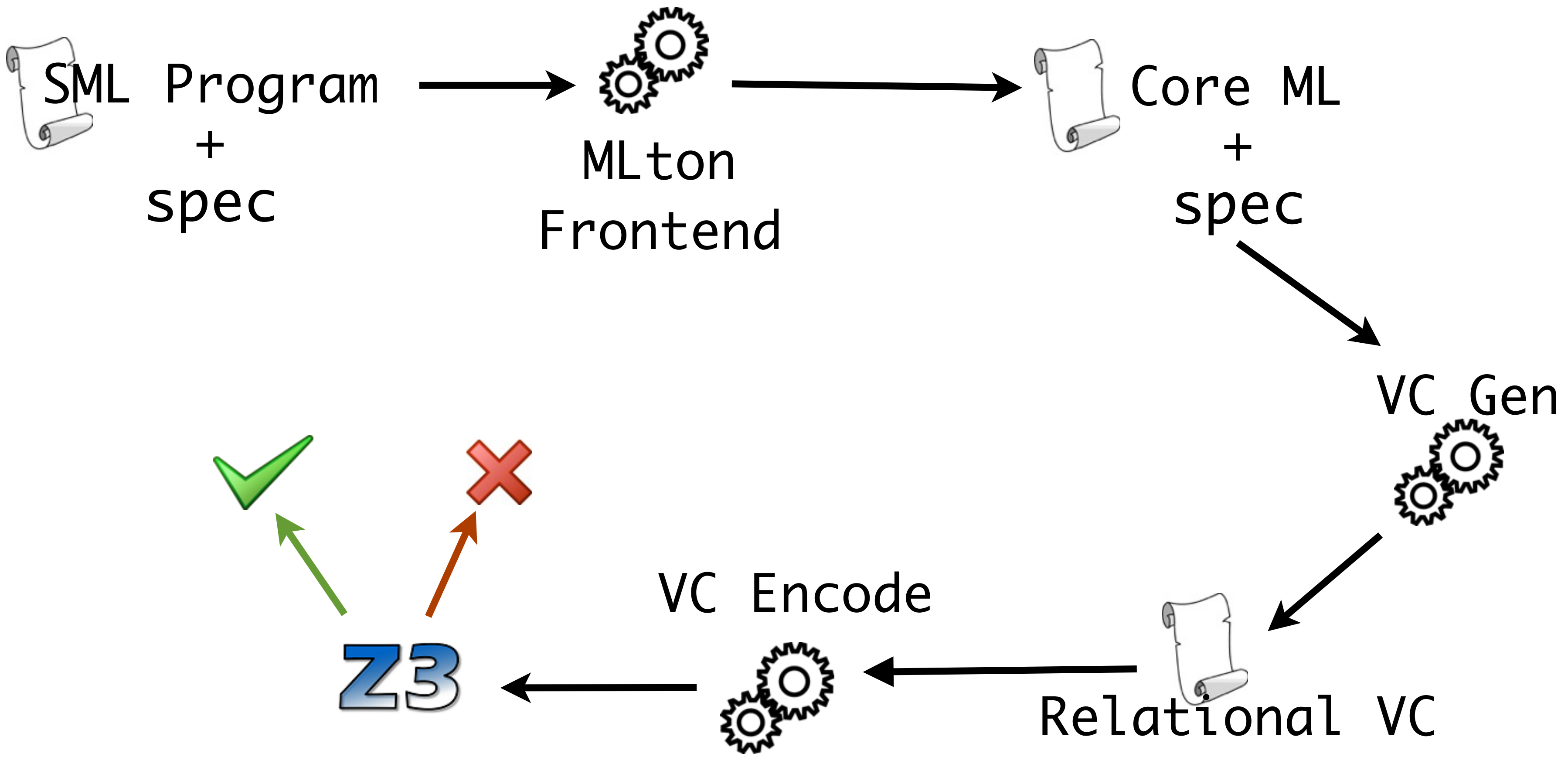
VC Encode

Z3

CATALYST

Implemented as extended type checking pass in MLton Standard ML compiler



SML Program
+
spec

MLton
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Core ML
+
spec

VC Gen

Relational VC

VC Encode

Z3

Validation

Lists	Okasaki trees	Functional Graphs	MLton functions
rev	inOrder	folds	alpha-rename
concat	preOrder	traversals	substitutions
map	postOrder	maps	SSA
foldl	treefoldl	⋮	
foldr	treefoldr		
exists	balance		
filter	rotate		
⋮	⋮		

Validation

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foldl	treefoldl	⋮	
foldr	treefoldr		
exists	balance		
filter	rotate		
⋮	⋮		

✓

Related Work

GADTs in OCaml and Haskell

Type refinements in F*

Abstract refinements in Liquid Types

Logical Relations

Shape analysis for higher-order control flow

Conclusions

Marriage of a relational specification language with a dependent type system capable of describing expressive structural invariants of functional data structures

Future Directions

- Extensions to deal with non-inductive structures
- Automated inference
- Basis for "lightweight" verified compilation

`https://github.com/tycon/catalyst`